Section A: Pure Mathematics

1 Show that, if k is a root of the quartic equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0\,, \qquad (*)$$

then k^{-1} is a root.

You are now given that a and b in $(*)$ are both real and are such that the roots are all real.

(i) Write down all the values of a and b for which $(*)$ has only one distinct root.

(ii) Given that $(*)$ has exactly three distinct roots, show that either $b = 2a - 2$ or $b = -2a - 2$.

(iii) Solve $(*)$ in the case $b = 2a - 2$, giving your solutions in terms of a.

Given that a and b are both real and that the roots of $(*)$ are all real, find necessary and sufficient conditions, in terms of a and b, for $(*)$ to have exactly three distinct real roots.

2 A function $\mathrm{f}(x)$ is said to be *concave* for $a < x < b$ if

$$t\,\mathrm{f}(x_1) + (1-t)\,\mathrm{f}(x_2) \leqslant \mathrm{f}\big(tx_1 + (1-t)x_2\big)\,,$$

for $a < x_1 < b$, $a < x_2 < b$ and $0 \leqslant t \leqslant 1$.

Illustrate this definition by means of a sketch, showing the chord joining the points $\big(x_1, \mathrm{f}(x_1)\big)$ and $\big(x_2, \mathrm{f}(x_2)\big)$, in the case $x_1 < x_2$ and $\mathrm{f}(x_1) < \mathrm{f}(x_2)$.

Explain why a function $\mathrm{f}(x)$ satisfying $\mathrm{f}''(x) < 0$ for $a < x < b$ is concave for $a < x < b$.

(i) By choosing t, x_1 and x_2 suitably, show that, if $\mathrm{f}(x)$ is concave for $a < x < b$, then

$$\mathrm{f}\Big(\frac{u+v+w}{3}\Big) \geqslant \frac{\mathrm{f}(u) + \mathrm{f}(v) + \mathrm{f}(w)}{3}\,,$$

for $a < u < b$, $a < v < b$ and $a < w < b$.

(ii) Show that, if A, B and C are the angles of a triangle, then

$$\sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{2}\,.$$

(iii) By considering $\ln(\sin x)$, show that, if A, B and C are the angles of a triangle, then

$$\sin A \times \sin B \times \sin C \leqslant \frac{3\sqrt{3}}{8}\,.$$

3 **(i)** Let

$$f(x) = \frac{1}{1 + \tan x}$$

for $0 \leqslant x < \frac{1}{2}\pi$.

Show that $f'(x) = -\dfrac{1}{1 + \sin 2x}$ and hence find the range of $f'(x)$.

Sketch the curve $y = f(x)$.

(ii) The function $g(x)$ is continuous for $-1 \leqslant x \leqslant 1$.

Show that the curve $y = g(x)$ has rotational symmetry of order 2 about the point (a, b) on the curve if and only if

$$g(x) + g(2a - x) = 2b\,.$$

Given that the curve $y = g(x)$ passes through the origin and has rotational symmetry of order 2 about the origin, write down the value of

$$\int_{-1}^{1} g(x)\, dx\,.$$

(iii) Show that the curve $y = \dfrac{1}{1 + \tan^k x}$, where k is a positive constant and $0 < x < \frac{1}{2}\pi$, has rotational symmetry of order 2 about a certain point (which you should specify) and evaluate

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \tan^k x}\, dx\,.$$

4 In this question, you may use the following identity without proof:

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B)\, \cos \tfrac{1}{2}(A - B)\,.$$

(i) Given that $0 \leqslant x \leqslant 2\pi$, find all the values of x that satisfy the equation

$$\cos x + 3\cos 2x + 3\cos 3x + \cos 4x = 0\,.$$

(ii) Given that $0 \leqslant x \leqslant \pi$ and $0 \leqslant y \leqslant \pi$ and that

$$\cos(x + y) + \cos(x - y) - \cos 2x = 1\,,$$

show that either $x = y$ or x takes one specific value which you should find.

(iii) Given that $0 \leqslant x \leqslant \pi$ and $0 \leqslant y \leqslant \pi$, find the values of x and y that satisfy the equation

$$\cos x + \cos y - \cos(x + y) = \tfrac{3}{2}\,.$$

5 In this question, you should ignore issues of convergence.

(i) Write down the binomial expansion, for $|x| < 1$, of $\dfrac{1}{1+x}$ and deduce that

$$\ln(1+x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$$

for $|x| < 1$.

(ii) Write down the series expansion in powers of x of e^{-ax}. Use this expansion to show that

$$\int_0^{\infty} \frac{(1-\mathrm{e}^{-ax})\,\mathrm{e}^{-x}}{x}\,\mathrm{d}x = \ln(1+a) \qquad (|a| < 1)\,.$$

(iii) Deduce the value of

$$\int_0^1 \frac{x^p - x^q}{\ln x}\,\mathrm{d}x \qquad (|p| < 1,\ |q| < 1)\,.$$

6 **(i)** Find all pairs of positive integers (n, p), where p is a prime number, that satisfy

$$n! + 5 = p\,.$$

(ii) In this part of the question you may use the following two theorems:

1. For $n \geqslant 7$, $1! \times 3! \times \cdots \times (2n-1)! > (4n)!$.
2. For every positive integer n, there is a prime number between $2n$ and $4n$.

Find all pairs of positive integers (n, m) that satisfy

$$1! \times 3! \times \cdots \times (2n-1)! = m!\,.$$

7 The points O, A and B are the vertices of an acute-angled triangle. The points M and N lie on the sides OA and OB respectively, and the lines AN and BM intersect at Q. The position vector of A with respect to O is $\mathbf{a}$, and the position vectors of the other points are labelled similarly.

Given that $|MQ| = \mu|QB|$, and that $|NQ| = \nu|QA|$, where μ and ν are positive and $\mu\nu < 1$, show that

$$\mathbf{m} = \frac{(1+\mu)\nu}{1+\nu}\,\mathbf{a}\,.$$

The point L lies on the side OB, and $|OL| = \lambda|OB|$. Given that ML is parallel to AN, express λ in terms of μ and ν.

What is the geometrical significance of the condition $\mu\nu < 1$?

8 **(i)** Use the substitution $v = \sqrt{y}$ to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha y^{\frac{1}{2}} - \beta y \qquad (y \geqslant 0, \quad t \geqslant 0)\,,$$

where α and β are positive constants. Find the non-constant solution $y_1(x)$ that satisfies $y_1(0) = 0$.

(ii) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha y^{\frac{2}{3}} - \beta y \qquad (y \geqslant 0, \quad t \geqslant 0)\,,$$

where α and β are positive constants. Find the non-constant solution $y_2(x)$ that satisfies $y_2(0) = 0$.

(iii) In the case $\alpha = \beta$, sketch $y_1(x)$ and $y_2(x)$ on the same axes, indicating clearly which is $y_1(x)$ and which is $y_2(x)$. You should explain how you determined the positions of the curves relative to each other.

Section B: Mechanics

9 Two small beads, A and B, of the same mass, are threaded onto a vertical wire on which they slide without friction, and which is fixed to the ground at P. They are released simultaneously from rest, A from a height of $8h$ above P and B from a height of $17h$ above P.

When A reaches the ground for the first time, it is moving with speed V. It then rebounds with coefficient of restitution $\frac{1}{2}$ and subsequently collides with B at height H above P.

Show that $H = \frac{15}{8}h$ and find, in terms of g and h, the speeds u_A and u_B of the two beads just before the collision.

When A reaches the ground for the second time, it is again moving with speed V. Determine the coefficient of restitution between the two beads.

10 A uniform elastic string lies on a smooth horizontal table. One end of the string is attached to a fixed peg, and the other end is pulled at constant speed u. At time $t = 0$, the string is taut and its length is a. Obtain an expression for the speed, at time t, of the point on the string which is a distance x from the peg at time t.

An ant walks along the string starting at $t = 0$ at the peg. The ant walks at constant speed v along the string (so that its speed relative to the peg is the sum of v and the speed of the point on the string beneath the ant). At time t, the ant is a distance x from the peg. Write down a first order differential equation for x, and verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{a+ut}\right) = \frac{v}{a+ut}\,.$$

Show that the time T taken for the ant to reach the end of the string is given by

$$uT = a(\mathrm{e}^k - 1)\,,$$

where $k = u/v$.

On reaching the end of the string, the ant turns round and walks back to the peg. Find in terms of T and k the time taken for the journey back.

11 The axles of the wheels of a motorbike of mass m are a distance b apart. Its centre of mass is a horizontal distance of d from the front axle, where $d < b$, and a vertical distance h above the road, which is horizontal and straight. The engine is connected to the rear wheel. The coefficient of friction between the ground and the rear wheel is μ, where $\mu < b/h$, and the front wheel is smooth.

You may assume that the sum of the moments of the forces acting on the motorbike about the centre of mass is zero. By taking moments about the centre of mass show that, as the acceleration of the motorbike increases from zero, the rear wheel will slip before the front wheel loses contact with the road if

$$\mu < \frac{b-d}{h} \,. \qquad (*)$$

If the inequality $(*)$ holds and the rear wheel does not slip, show that the maximum acceleration is

$$\frac{\mu d g}{b - \mu h} \,.$$

If the inequality $(*)$ does not hold, find the maximum acceleration given that the front wheel remains in contact with the road.

Section C: Probability and Statistics

12 In a game, I toss a coin repeatedly. The probability, p, that the coin shows Heads on any given toss is given by

$$p = \frac{N}{N+1},$$

where N is a positive integer. The outcomes of any two tosses are independent.

The game has two versions. In each version, I can choose to stop playing after any number of tosses, in which case I win £H, where H is the number of Heads I have tossed. However, the game may end before that, in which case I win nothing.

(i) In version 1, the game ends when the coin first shows Tails (if I haven't stopped playing before that).

I decide from the start to toss the coin until a total of h Heads have been shown, unless the game ends before then. Find, in terms of h and p, an expression for my expected winnings and show that I can maximise my expected winnings by choosing $h = N$.

(ii) In version 2, the game ends when the coin shows Tails on two *consecutive* tosses (if I haven't stopped playing before that).

I decide from the start to toss the coin until a total of h Heads have been shown, unless the game ends before then. Show that my expected winnings are

$$\frac{hN^h(N+2)^h}{(N+1)^{2h}}.$$

In the case $N = 2$, use the approximation $\log_3 2 \approx 0.63$ to show that the maximum value of my expected winnings is approximately £3.

13 Four children, A, B, C and D, are playing a version of the game 'pass the parcel'. They stand in a circle, so that $ABCDA$ is the clockwise order. Each time a whistle is blown, the child holding the parcel is supposed to pass the parcel immediately exactly one place clockwise. In fact each child, independently of any other past event, passes the parcel clockwise with probability $\frac{1}{4}$, passes it anticlockwise with probability $\frac{1}{4}$ and fails to pass it at all with probability $\frac{1}{2}$. At the start of the game, child A is holding the parcel.

The probability that child A is holding the parcel just after the whistle has been blown for the nth time is A_n, and B_n, C_n and D_n are defined similarly.

(i) Find A_1, B_1, C_1 and D_1. Find also A_2, B_2, C_2 and D_2.

(ii) By first considering $B_{n+1} + D_{n+1}$, or otherwise, find B_n and D_n.

Find also expressions for A_n and C_n in terms of n.