Section A: Pure Mathematics

1 The function f is defined, for $x \neq 1$ and $x \neq 2$ by

$$\mathrm{f}(x) = \frac{1}{(x-1)(x-2)}$$

Show that for $|x| < 1$

$$\mathrm{f}(x) = \sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

and that for $1 < |x| < 2$

$$\mathrm{f}(x) = -\sum_{n=1}^{\infty} x^{-n} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

Find an expression for f(x) which is valid for $|x| > 2$.

2 The numbers x, y and z are non-zero, and satisfy

$$2a - 3y = \frac{(z-x)^2}{y} \quad \text{and} \quad 2a - 3z = \frac{(x-y)^2}{z},$$

for some number a. If $y \neq z$, prove that

$$x + y + z = a,$$

and that

$$2a - 3x = \frac{(y-z)^2}{x}.$$

Determine whether this last equation holds *only* if $y \neq z$.

3 The quadratic equation $x^2 + bx + c = 0$, where b and c are real, has the properly that if k is a (possibly complex) root, then k^{-1} is a root. Determine carefully the restriction that this property places on b and c. If, in addition to this property, the equation has the further property that if k is a root, then $1 - k$ is a root, find b and c.

Show that

$$x^3 - \tfrac{3}{2}x^2 - \tfrac{3}{2}x + 1 = 0$$

is the only cubic equation of the form $x^3 + px^2 + qx + r = 0$, where p, q and r are real, which has both the above properties.

4 The complex number w is such that $w^2 - 2x$ is real.

(i) Sketch the locus of w in the Argand diagram.

(ii) If $w^2 = x + \mathrm{i}y$, describe fully and sketch the locus of points (x, y) in the x-y plane.

The complex number t is such that $t^2 - 2t$ is imaginary. If $t^2 = p + \mathrm{i}q$, sketch the locus of points (p, q) in the p-q plane.

5 By considering the imaginary part of the equation $z^7 = 1$, or otherwise, find all the roots of the equation

$$t^6 - 21t^4 + 35t^2 - 7 = 0.$$

You should justify each step carefully.

Hence, or otherwise, prove that

$$\tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{6\pi}{7} = \sqrt{7}.$$

Find the corresponding result for

$$\tan\frac{2\pi}{n}\tan\frac{4\pi}{n}\cdots\tan\frac{(n-1)\pi}{n}$$

in the two cases $n = 9$ and $n = 11$.

6 Show that the following functions are positive when x is positive:

(i) $x - \tanh x$

(ii) $x \sinh x - 2\cosh x + 2$

(iii) $2x\cosh 2x - 3\sinh 2x + 4x$.

The function f is defined for $x > 0$ by

$$\mathrm{f}(x) = \frac{x(\cosh x)^{\frac{1}{3}}}{\sinh x}.$$

Show that $\mathrm{f}(x)$ has no turning points when $x > 0$, and sketch $\mathrm{f}(x)$ for $x > 0$.

7 The integral I is defined by

$$I = \int_1^2 \frac{(2-2x+x^2)^k}{x^{k+1}}\,\mathrm{d}x$$

where k is a constant. Show that

$$I = \int_0^1 \frac{(1+x^2)^k}{(1+x)^{k+1}}\,\mathrm{d}x = \int_0^{\frac14\pi} \frac{\mathrm{d}\theta}{\left[\sqrt{2}\cos\theta\cos\left(\frac14\pi-\theta\right)\right]^{k+1}} = 2\int_0^{\frac18\pi} \frac{\mathrm{d}\theta}{\left[\sqrt{2}\cos\theta\cos\left(\frac14\pi-\theta\right)\right]^{k+1}}.$$

Hence show that

$$I = 2\int_0^{\sqrt{2}-1} \frac{(1+x^2)^k}{(1+x)^{k+1}}\,\mathrm{d}x$$

Deduce that

$$\int_1^{\sqrt{2}} \left(\frac{2-2x^2+x^4}{x^2}\right)^k \frac{1}{x}\,\mathrm{d}x = \int_1^{\sqrt{2}} \left(\frac{2-2x+x^2}{x}\right)^k \frac{1}{x}\,\mathrm{d}x$$

8 In a crude model of population dynamics of a community of aardvarks and buffaloes, it is assumed that, if the numbers of aardvarks and buffaloes in any year are A and B respectively, then the numbers in the following year at $\frac14 A + \frac34 B$ and $\frac32 B - \frac12 A$ respectively. It does not matter if the model predicts fractions of animals, but a non-positive number of buffaloes means that the species has become extinct, and the model ceases to apply. Using matrices or otherwise, show that the ratio of the number of aardvarks to the number of buffaloes can remain the same each year, provided it takes one of two possible values.

Let these two possible values be x and y, and let the numbers of aardvarks and buffaloes in a given year be a and b respectively. By writing the vector (a, b) as a linear combination of the vectors $(x, 1)$ and $(y, 1)$, or otherwise, show how the numbers of aardvarks and buffaloes in subsequent years may be found. On a sketch of the a-b plane, mark the regions which correspond to the following situations

(i) an equilibrium population is reached as time $t \to \infty$;

(ii) buffaloes become extinct after a finite time;

(iii) buffaloes approach extinction as $t \to \infty$.

9 Give a careful argument to show that, if G_1 and G_2 are subgroups of a finite group G such that every element of G is either in G_1 or in G_2, then either $G_1 = G$ or $G_2 = G$.

Give an example of a group H which has three subgroups H_1, H_2 and H_3 such that every element of H is either in H_1, H_2 or H_3 and $H_1 \neq H, H_2 \neq H, H_3 \neq H$.

10 The surface S in 3-dimensional space is described by the equation

$$\mathbf{a} \cdot \mathbf{r} + ar = a^2,$$

where $\mathbf{r}$ is the position vector with respect to the origin O, $\mathbf{a}(\neq \mathbf{0})$ is the position vector of a fixed point, $r = |\mathbf{r}|$ and $a = |\mathbf{a}|$. Show, with the aid of a diagram, that S is the locus of points which are equidistant from the origin O and the plane $\mathbf{r} \cdot \mathbf{a} = a^2$.

The point P, with position vector $\mathbf{p}$, lies in S, and the line joining P to O meets S again at Q. Find the position vector of Q.

The line through O orthogonal to $\mathbf{p}$ and $\mathbf{a}$ meets S at T and T'. Show that the position vectors of T and T' are

$$\pm \frac{1}{\sqrt{2ap - a^2}} \mathbf{a} \times \mathbf{p},$$

where $p = |\mathbf{p}|$.

Show that the area of the triangle PQT is

$$\frac{ap^2}{2p - a}.$$

Section B: Mechanics

11 A heavy particle lies on a smooth horizontal table, and is attached to one end of a light inextensible string of length L. The other end of the string is attached to a point P on the circumference of the base of a vertical post which is fixed into the table. The base of the post is a circle of radius a with its centre at a point O on the table. Initially, at time $t = 0$, the string is taut and perpendicular to the line OP. The particle is then struck in such a way that the string starts winding round the post and remains taut. At a later time t, a length $a\theta(t)$ $(< L)$ of the string is in contact with the post. Using cartesian axes with origin O, find the position and velocity vectors of the particle at time t in terms of a, L, θ and $\dot{\theta}$, and hence show that the speed of the particle is $(L - a\theta)\dot{\theta}$.

If the initial speed of the particle is v, show that the particle hits the post at a time $L^2/(2av)$.

12 One end of a thin uniform inextensible, but perfectly flexible, string of length l and uniform mass per unit length is held at a point on a smooth table a distance $d(< l)$ away from a small vertical hole in the surface of the table. The string passes through the hole so that a length $l - d$ of the string hangs vertically. The string is released from rest. Assuming that the height of the table is greater than l, find the time taken for the end of the string to reach the top of the hole.

13 A librarian wishes to pick up a row of identical books from a shelf, by pressing her hands on the outer covers of the two outermost books and lifting the whole row together. The covers of the books are all in parallel vertical planes, and the weight of each book is W. With each arm, the librarian can exert a maximum force of P in the vertical direction, and, independently, a maximum force of Q in the horizontal direction. The coefficient of friction between each pair of books and also between each hand and a book is μ. Derive an expression for the maximum number of books that can be picked up without slipping, using this method.

[You may assume that the books are thin enough for the rotational effect of the couple on each book to be ignored.]

14 Two particles of mass M and m $(M > m)$ are attached to the ends of a light rod of length $2l$. The rod is fixed at its midpoint to a point O on a horizontal axle so that the rod can swing freely about O in a vertical plane normal to the axle. The axle rotates about a *vertical* axis through O at a constant angular speed ω such that the rod makes a constant angle α $(0 < \alpha < \frac{1}{2}\pi)$ with the vertical. Show that

$$\omega^2 = \left(\frac{M-m}{M+m}\right)\frac{g}{l\cos\alpha}.$$

Show also that the force of reaction of the rod on the axle is inclined at an angle

$$\tan^{-1}\left[\left(\frac{M-m}{M+m}\right)^2\tan\alpha\right]$$

with the downward vertical.

Section C: Probability and Statistics

15 An examination consists of several papers, which are marked independently. The mark given for each paper can be an integer from 0 to m inclusive, and the total mark for the examination is the sum of the marks on the individual papers. In order to make the examination completely fair, the examiners decide to allocate the mark for each paper at random, so that the probability that any given candidate will be allocated k marks $(0 \leqslant k \leqslant m)$ for a given paper is $(m+1)^{-1}$. If there are just two papers, show that the probability that a given candidate will receive a total of n marks is

$$\frac{2m-n+1}{(m+1)^2}$$

for $m < n \leqslant 2m$, and find the corresponding result for $0 \leqslant n \leqslant m$.

If the examination consists of three papers, show that the probability that a given candidate will receive a total of n marks is

$$\frac{6mn-4m^2-2n^2+3m+2}{2\,(m+1)^2}$$

in the case $m < n \leqslant 2m$. Find the corresponding result for $0 \leqslant n \leqslant m$, and deduce the result for $2m < n \leqslant 3m$.

16 Find the probability that the quadratic equation

$$X^2 + 2BX + 1 = 0$$

has real roots when B is normally distributed with zero mean and unit variance.

Given that the two roots X_1 and X_2 are real, find:

(i) the probability that both X_1 and X_2 are greater than $\frac{1}{5}$;

(ii) the expected value of $|X_1 + X_2|$;

giving your answers to three significant figures.