Section A: Pure Mathematics

1 Sketch the graph of

$$y = \frac{x^2 \mathrm{e}^{-x}}{1+x},$$

for $-\infty < x < \infty$.

Show that the value of

$$\int_0^\infty \frac{x^2 \mathrm{e}^{-x}}{1+x} \, \mathrm{d}x$$

lies between 0 and 1.

2 The real numbers $u_0, u_1, u_2, \ldots$ satisfy the difference equation

$$a u_{n+2} + b u_{n+1} + c u_n = 0 \qquad (n = 0, 1, 2, \ldots),$$

where a, b and c are real numbers such that the quadratic equation

$$ax^2 + bx + c = 0$$

has two distinct real roots α and β. Show that the above difference equation is satisfied by the numbers u_n defined by

$$u_n = A\alpha^n + B\beta^n,$$

where

$$A = \frac{u_1 - \beta u_0}{\alpha - \beta} \qquad \text{and} \qquad B = \frac{u_1 - \alpha u_0}{\beta - \alpha}.$$

Show also, by induction, that these numbers provide the only solution.

Find the numbers v_n $(n = 0, 1, 2, \ldots)$ which satisfy

$$8(n+2)(n+1)v_{n+2} - 2(n+3)(n+1)v_{n+1} - (n+3)(n+2)v_n = 0$$

with $v_0 = 0$ and $v_1 = 1$.

3 Give a parametric form for the curve in the Argand diagram determined by $|z - \mathrm{i}| = 2$.

Let $w = (z + \mathrm{i})/(z - \mathrm{i})$. Find and sketch the locus, in the Argand diagram, of the point which represents the complex number w when

(i) $|z - \mathrm{i}| = 2$;

(ii) z is real;

(iii) z is imaginary.

4 A kingdom consists of a vast plane with a central parabolic hill. In a vertical cross-section through the centre of the hill, with the x-axis horizontal and the z-axis vertical, the surface of the plane and hill is given by

$$x = \begin{cases} \dfrac{1}{2a}(a^2 - x^2) & \text{for } |x| \leqslant a, \\ 0 & \text{for } |x| > a. \end{cases}$$

The whole surface is formed by rotating this cross-section about the z-axis. In the (x, z) plane through the centre of the hill, the king has a summer residence at $(-R, 0)$ and a winter residence at $(R, 0)$, where $R > a$. He wishes to connect them by a road, consisting of the following segments:

(i) a path in the (x, z) plane joining $(-R, 0)$ to $(-b, (a^2 - b^2)/2a)$, where $0 \leqslant b \leqslant a$.

(ii) a horizontal semicircular path joining the two points $(\pm b, (a^2 - b^2)/2a)$, if $b \neq 0$;

(iii) a path in the (x, z) plane joining $(b, (a^2 - b^2)/2a)$ to $(R, 0)$.

The king wants the road to be as short as possible. Advise him on his choice of b.

5 A firm of engineers obtains the right to dig and exploit an undersea tunnel. Each day the firm borrows enough money to pay for the day's digging, which costs $£c$, and to pay the daily interest of $100k\%$ on the sum already borrowed. The tunnel takes T days to build, and, once finished, earns $£d$ a day, all of which goes to pay the daily interest and repay the debt until it is fully paid. The financial transactions take place at the end of each day's work. Show that S_n, the total amount borrowed by the end of day n, is given by

$$S_n = \frac{c[(1 + k)^n - 1]}{k}$$

for $n \leqslant T$.

Given that $S_{T+m} > 0$, where $m > 0$, express S_{T+m} in terms of c, d, k, T and m.

Show that, if $d/c > (1 + k)^T - 1$, the firm will eventually pay off the debt.

6 Let $\mathrm{f}(x) = \sin 2x \cos x$. Find the 1988th derivative of $\mathrm{f}(x)$.

Show that the smallest positive value of x for which this derivative is zero is $\frac{1}{3}\pi + \epsilon$, where ϵ is approximately equal to

$$\frac{3^{-1988}\sqrt{3}}{2}.$$

7 For $n = 0, 1, 2, \ldots$, the functions y_n satisfy the differential equation

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} - \omega^2 x^2 y_n = -(2n+1)\omega y_n,$$

where ω is a positive constant, and $y_n \to 0$ and $\mathrm{d}y_n/\mathrm{d}x \to 0$ as $x \to +\infty$ and as $x \to -\infty$. Verify that these conditions are satisfied, for $n = 0$ and $n = 1$, by

$$y_0(x) = \mathrm{e}^{-\lambda x^2} \qquad \text{and} \qquad y_1(x) = x\mathrm{e}^{-\lambda x^2}$$

for some constant λ, to be determined.

Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y_m \frac{\mathrm{d}y_n}{\mathrm{d}x} - y_n \frac{\mathrm{d}y_m}{\mathrm{d}x}\right) = 2(m-n)\omega y_m y_n,$$

and deduce that, if $m \neq n$,

$$\int_{-\infty}^{\infty} y_m(x) y_n(x)\,\mathrm{d}x = 0.$$

8 Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

For $i = 1, 2$, and 3, let P_i be the point $(at_i^2, 2at_i)$, where t_1, t_2 and t_3 are all distinct. Let A_1 be the area of the triangle formed by the tangents at P_1, P_2 and P_3, and let A_2 be the area of the triangle formed by the normals at P_1, P_2 and P_3. Using the fact that the area of the triangle with vertices at $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is the absolute value of

$$\tfrac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix},$$

show that $A_3 = (t_1 + t_2 + t_3)^2 A_1$.

Deduce a necessary and sufficient condition in terms of t_1, t_2 and t_3 for the normals at P_1, P_2 and P_3 to be concurrent.

9 Let G be a finite group with identity e. For each element $g \in G$, the order of g, $o(g)$, is defined to be the smallest positive integer n for which $g^n = e$.

(i) Show that, if $o(g) = n$ and $g^N = e$, then n divides N.

(ii) Let g and h be elements of G. Prove that, for any integer m,

$$gh^m g^{-1} = (ghg^{-1})^m.$$

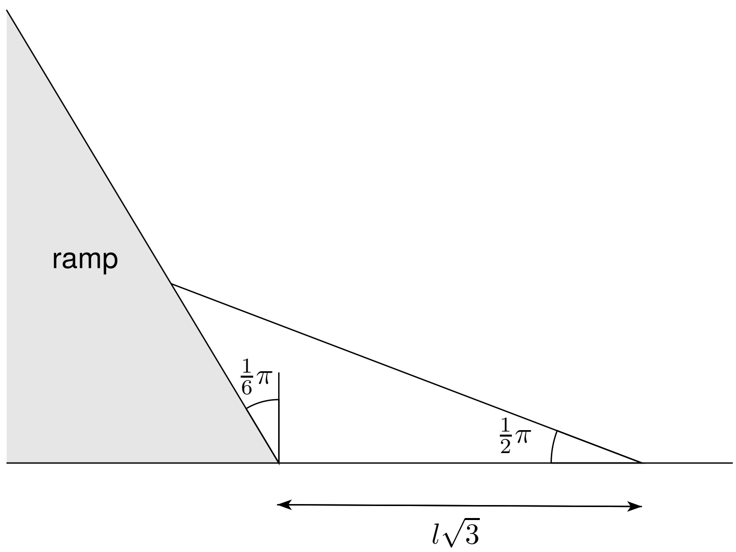
(iii) Let g and h be elements of G, such that $g^5 = e, h \neq e$ and $ghg^{-1} = h^2$. Prove that $g^2 h g^{-2} = h^4$ and find $o(h)$.

10 Four greyhounds A, B, C and D are held at positions such that $ABCD$ is a large square. At a given instant, the dogs are released and A runs directly towards B at constant speed v, B runs directly towards C at constant speed v, and so on. Show that A's path is given in polar coordinates (referred to an origin at the centre of the field and a suitable initial line) by $r = \lambda e^{-\theta}$, where λ is a constant.

Generalise this result to the case of n dogs held at the vertices of a regular n-gon ($n \geqslant 3$).

Section B: Mechanics

11 A uniform ladder of length l and mass m rests with one end in contact with a smooth ramp inclined at an angle of $\pi/6$ to the vertical. The foot of the ladder rests, on horizontal ground, at a distance $l/\sqrt{3}$ from the foot of the ramp, and the coefficient of friction between the ladder and the ground is μ. The ladder is inclined at an angle $\pi/6$ to the horizontal, in the vertical plane containing a line of greatest slope of the ramp. A labourer of mass m intends to climb slowly to the top of the ladder.



(i) Find the value of μ if the ladder slips as soon as the labourer reaches the midpoint.

(ii) Find the minimum value of μ which will ensure that the labourer can reach the top of the ladder.

12 A smooth billiard ball moving on a smooth horizontal table strikes another identical ball which is at rest. The coefficient of restitution between the balls is $e(< 1)$. Show that after the collision the angle between the velocities of the balls is less than $\frac{1}{2}\pi$.

Show also that the maximum angle of deflection of the first ball is

$$\sin^{-1}\left(\frac{1+e}{3-e}\right).$$

13 A goalkeeper stands on the goal-line and kicks the football directly into the wind, at an angle α to the horizontal. The ball has mass m and is kicked with velocity $\mathbf{v}_0$. The wind blows horizontally with constant velocity $\mathbf{w}$ and the air resistance on the ball is mk times its velocity relative to the wind velocity, where k is a positive constant. Show that the equation of motion of the ball can be written in the form

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + k\mathbf{v} = \mathbf{g} + k\mathbf{w},$$

where $\mathbf{v}$ is the ball's velocity relative to the ground, and $\mathbf{g}$ is the acceleration due to gravity.

By writing down horizontal and vertical equations of motion for the ball, or otherwise, find its position at time t after it was kicked.

On the assumption that the goalkeeper moves out of the way, show that if $\tan\alpha = |\mathbf{g}| / (k\,|\mathbf{w}|)$, then the goalkeeper scores an own goal.

14 A small heavy bead can slide smoothly in a vertical plane on a fixed wire with equation

$$y = x - \frac{x^2}{4a},$$

where the y-axis points vertically upwards and a is a positive constant. The bead is projected from the origin with initial speed V along the wire.

(i) Show that for a suitable value of V, to be determined, a motion is possible throughout which the bead exerts no pressure on the wire.

(ii) Show that θ, the angle between the particle's velocity at time t and the x-axis, satisfies

$$\frac{4a^2\dot{\theta}^2}{\cos^6\theta} + 2ga(1 - \tan^2\theta) = V^2.$$

Section C: Probability and Statistics

15 Each day, books returned to a library are placed on a shelf in order of arrival, and left there. When a book arrives for which there is no room on the shelf, that book and all books subsequently returned are put on a trolley. At the end of each day, the shelf and trolley are cleared. There are just two-sizes of book: thick, requiring two units of shelf space; and thin, requiring one unit. The probability that a returned book is thick is p, and the probability that it is thin is $q = 1 - p$. Let $M(n)$ be the expected number of books that will be put on the shelf, when the length of the shelf is n units and n is an integer, on the assumption that more books will be returned each day than can be placed on the shelf. Show, giving reasoning, that

(i) $M(0) = 0$;

(ii) $M(1) = q$;

(iii) $M(n) - qM(n-1) - pM(n-2) = 1$, for $n \geqslant 2$.

Verify that a possible solution to these equations is

$$M(n) = A(-p)^n + B + Cn,$$

where A, B and C are numbers independent of n which you should express in terms of p.

16 Balls are chosen at random without replacement from an urn originally containing m red balls and $M-m$ green balls. Find the probability that exactly k red balls will be chosen in n choices $(0 \leqslant k \leqslant m, 0 \leqslant n \leqslant M)$.

The random variables X_i $(i = 1, 2, \ldots, n)$ are defined for $n \leqslant M$ by

$$X_i = \begin{cases} 0 & \text{if the } i\text{th ball chosen is green} \\ 1 & \text{if the } i\text{th ball chosen is red.} \end{cases}$$

Show that

(i) $\mathrm{P}(X_i = 1) = \dfrac{m}{M}$.

(ii) $\mathrm{P}(X_i = 1 \text{ and } X_j = 1) = \dfrac{m(m-1)}{M(M-1)}$, for $i \neq j$.

Find the mean and variance of the random variable X defined by

$$X = \sum_{i=1}^{n} X_i.$$