Section A: Pure Mathematics

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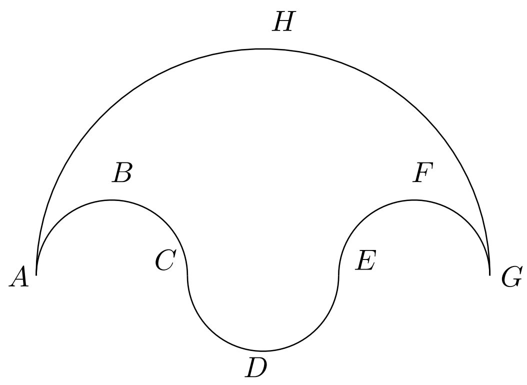

In the above diagram, ABC, CDE, EFG and AHG are semicircles and A, C, E, G lie on a straight line. The radii of ABC, EFG, AHG are h, h and r respectively, where $2h < r$. Show that the area enclosed by $ABCDEFGH$ is equal to that of a circle with diameter HD.

Each semicircle is now replaced by a portion of a parabola, with vertex at the midpoint of the semicircle arc, passing through the endpoints (so, for example, ABC is replaced by part of a parabola passing through A and C and with vertex at B). Find a formula in terms of r and h for the area enclosed by $ABCDEFGH$.

2 For $x > 0$ find $\int x \ln x \, \mathrm{d}x$.

By approximating the area corresponding to $\int_0^1 x \ln(1/x) \, \mathrm{d}x$ by n rectangles of equal width and with their top right-hand vertices on the curve $y = x \ln(1/x)$, show that, as $n \to \infty$,

$$\frac{1}{2}\left(1 + \frac{1}{n}\right) \ln n - \frac{1}{n^2}\left[\ln\left(\frac{n!}{0!}\right) + \ln\left(\frac{n!}{1!}\right) + \ln\left(\frac{n!}{2!}\right) + \cdots + \ln\left(\frac{n!}{(n-1)!}\right)\right] \to \frac{1}{4}.$$

[You may assume that $x \ln x \to 0$ as $x \to 0$.]

3 In the triangle OAB, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $OA = OB = 1$. Points C and D trisect AB (i.e. $AC = CD = DB = \frac{1}{3}AB$). X and Y lie on the line-segments OA and OB respectively, in such a way that CY and DX are perpendicular, and $OX + OY = 1$. Denoting OX by x, obtain a condition relating x and $\mathbf{a}\cdot\mathbf{b}$, and prove that

$$\frac{8}{17} \leqslant \mathbf{a}\cdot\mathbf{b} \leqslant 1.$$

If the angle AOB is as large as possible, determine the distance OE, where E is the point of intersection of CY and DX.

4 Six points A, B, C, D, E and F lie in three dimensional space and are in general positions, that is, no three are collinear and no four lie on a plane. All possible line segments joining pairs of points are drawn and coloured either gold or silver. Prove that there is a triangle whose edges are entirely of one colour. [**Hint**: consider segments radiating from A.]

Give a sketch showing that the result is false for five points in general positions.

5 Write down the binomial expansion of $(1+x)^n$, where n is a positive integer.

(i) By substituting particular values of x in the above expression, or otherwise, show that, if n is an even positive integer,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{n} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots + \binom{n}{n-1} = 2^{n-1}.$$

(ii) Show that, if n is any positive integer, then

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

Hence evaluate

$$\sum_{r=0}^{n} \left(r + (-1)^r\right)\binom{n}{r}.$$

6 The normal to the curve $y = \mathrm{f}(x)$ at the point P with coordinates $(x, \mathrm{f}(x))$ cuts the y-axis at the point Q. Derive an expression in terms of x, $\mathrm{f}(x)$ and $\mathrm{f}'(x)$ for the y-coordinate of Q.

If, for all x, $PQ = \sqrt{\mathrm{e}^{x^2} + x^2}$, find a differential equation satisfied by $\mathrm{f}(x)$. If the curve also has a minimum point $(0, -2)$, find its equation.

7 Sketch the curve $y^2 = 1 - |x|$. A rectangle, with sides parallel to the axes, is inscribed within this curve. Show that the largest possible area of the rectangle is $8/\sqrt{27}$.

Find the maximum area of a rectangle similarly inscribed within the curve given by $y^{2m} = (1 - |x|)^n$, where m and n are positive integers, with n odd.

8 By using de Moivre's theorem, or otherwise, show that

(i) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$;

(ii) $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

Hence, or otherwise, find all the real roots of the equation

$$16x^6 - 28x^4 + 13x^2 - 1 = 0.$$

[No credit will be given for numerical approximations.]

9 Sketch the graph of $8y = x^3 - 12x$ for $-4 \leqslant x \leqslant 4$, marking the coordinates of the turning points. Similarly marking the turning points, sketch the corresponding graphs in the (X, Y)-plane, if

$$\begin{array}{lll} \text{(a)} & X = \frac{1}{2}x, & Y = y, \\ \text{(b)} & X = x, & Y = \frac{1}{2}y, \\ \text{(c)} & X = \frac{1}{2}x + 1, & Y = y, \\ \text{(d)} & X = x, & Y = \frac{1}{2}y + 1. \end{array}$$

Find values for a, b, c, d such that, if $X = ax + b$, $Y = cy + d$, then the graph in the (X, Y)-plane corresponding to $8y = x^3 - 12x$ has turning points at $(X, Y) = (0, 0)$ and $(X, Y) = (1, 1)$.

Section B: Mechanics

10 A spaceship of mass M is travelling at constant speed V in a straight line when it enters a force field which applies a resistive force acting directly backwards and of magnitude $M\omega(v^2 + V^2)/v$, where v is the instantaneous speed of the spaceship, and ω is a positive constant. No other forces act on the spaceship. Find the distance travelled from the edge of the force field until the speed is reduced to $\frac{1}{2}V$.

As soon as the spaceship has travelled this distance within the force field, the field is altered to a constant resistive force, acting directly backwards, whose magnitude is within 10% of that of the force acting on the spaceship immediately before the change. If z is the extra distance travelled by the spaceship before coming instantaneously to rest, determine limits between which z must lie.

11 A shot-putter projects a shot at an angle θ above the horizontal, releasing it at height h above the level ground, with speed v. Show that the distance R travelled horizontally by the shot from its point of release until it strikes the ground is given by

$$R = \frac{v^2}{2g}\sin 2\theta\left(1 + \sqrt{1 + \frac{2hg}{v^2\sin^2\theta}}\right).$$

The shot-putter's style is such that currently $\theta = 45°$. Determine (with justification) whether a small decrease in θ will increase R.

[Air resistance may be neglected.]

12 A regular tetrahedron $ABCD$ of mass M is made of 6 identical uniform rigid rods, each of length $2a$. Four light elastic strings XA, XB, XC and XD, each of natural length a and modulus of elasticity λ, are fastened together at X, the other end of each string being attached to the corresponding vertex. Given that X lies at the centre of mass of the tetrahedron, find the tension in each string.

The tetrahedron is at rest on a smooth horizontal table, with B, C and D touching the table, and the ends of the strings at X attached to a point O fixed in space. Initially the centre of mass of the tetrahedron coincides with O. Suddenly the string XA breaks, and the tetrahedron as a result rises vertically off the table. If the maximum height subsequently attained is such that BCD is level with the fixed point O, show that (to 2 significant figures)

$$\frac{Mg}{\lambda} = 0.098.$$

13 A uniform ladder of mass M rests with its upper end against a smooth vertical wall, and with its lower end on a rough slope which rises upwards towards the wall and makes an angle of ϕ with the horizontal. The acute angle between the ladder and the wall is θ. If the ladder is in equilibrium, show that N and F, the normal reaction and frictional force at the foot of the ladder are given by

$$N = Mg\left(\cos\phi - \frac{\tan\theta\sin\phi}{2}\right),$$

$$F = Mg\left(\sin\phi + \frac{\tan\theta\cos\phi}{2}\right).$$

If the coefficient of friction between the ladder and the slope is 2, and $\phi = 45^\circ$, what is the largest value of θ for which the ladder can rest in equilibrium?

Section C: Probability and Statistics

14 The prevailing winds blow in a constant southerly direction from an enchanted castle. Each year, according to an ancient tradiction, a princess releases 96 magic seeds from the castle, which are carried south by the wind before falling to rest. South of the castle lies one league of grassy parkland, then one league of lake, then one league of farmland, and finally the sea. If a seed falls on land it will immediately grow into a fever tree. (Fever trees do not grow in water). Seeds are blown independently of each other. The random variable L is the distance in leagues south of the castle at which a seed falls to rest (either on land or water). It is known that the probability density function f of L is given by

$$\mathrm{f}(x) = \begin{cases} \frac{1}{2} - \frac{1}{8}x & \text{for } 0 \leqslant x \leqslant 4, \\ 0 & \text{otherwise.} \end{cases}$$

What is the mean number of fever trees which begin to grow each year?

(i) The random variable Y is defined as the distance in leagues south of the castle at which a new fever tree grows from a seed carried by the wind. Sketch the probability density function of Y, and find the mean of Y.

(ii) One year messengers bring the king the news that 23 new fever trees have grown in the farmland. The wind never varies, and so the king suspects that the ancient tradition have not been followed properly. Is he justified in his suspicions?

15 I can choose one of three routes to cycle to school. Via Angle Avenue the distance is 5 km, and I am held up at a level crossing for A minutes, where A is a continuous random variable uniformly distributed between 0 and 10. Via Bend Boulevard the distance is 4 km, and I am delayed, by talking to each of B friends for 3 minutes, for a total of $3B$ minutes, where B is a random variable whose distribution is Poisson with mean 4. Via Detour Drive the distance should be only 2 km, but in addition, due to never-ending road works, there are five places at each of which, with probability $\frac{4}{5}$, I have to make a detour that increases the distance by 1 km. Except when delayed by talking to friends or at the level crossing, I cycle at a steady 12 km h^{-1}. For each of the three routs, calculate the probability that a journey lasts at least 27 minutes.

Each day I choose one of the three routes at random, and I am equally likely to choose any of the three alternatives. One day I arrive at school after a journey of at least 27 minutes. What is the probability that I came via Bend Boulevard?

Which route should I use all the time:

(i) if I wish my average journey time to be as small as possible;

(ii) if I wish my journey time to be less than 32 minutes as often as possible?

Justify your answers.

16 A and B play a guessing game. Each simultaneously names one of the numbers $1, 2, 3$. If the numbers differ by 2, whoever guessed the *smaller* pays the opponent $£2$. If the numbers differ by 1, whoever guessed the *larger* pays the opponent $£1$. Otherwise no money changes hands. Many rounds of the game are played.

(i) If A says he will always guess the same number N, explain (for each value of N) how B can maximise his winnings.

(ii) In an attempt to improve his play, A announces that he will guess each number at random with probability $\frac{1}{3}$, guesses on different rounds being independent. To counter this, B secretly decides to guess j with probability b_j ($j = 1, 2, 3,\ b_1 + b_2 + b_3 = 1$), guesses on different rounds being independent. Derive an expression for B's expected winnings on any round. How should the probabilities b_j be chosen so as to maximize this expression?

(iii) A now announces that he will guess j with probability a_j ($j = 1, 2, 3,\ a_1 + a_2 + a_3 = 1$). If B guesses j with probability b_j ($j = 1, 2, 3,\ b_1 + b_2 + b_3 = 1$), obtain an expression for his expected winnings in the form

$$Xa_1 + Ya_2 + Za_3.$$

Show that he can choose b_1, b_2 and b_3 such that X, Y and Z are all non-negative. Deduce that, whatever values for a_j are chosen by A, B can ensure that in the long run he loses no money.