Section A: Pure Mathematics

1 Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Show how the cubic equation

$$24x^3 - 72x^2 + 66x - 19 = 0 \qquad (*)$$

can be reduced to the form

$$4z^3 - 3z = k$$

by means of the substitution $y = x + a$ and $z = by$, for suitable values of the constants a and b. Hence find the three roots of the equation $(*)$, to three significant figures.

Show, by means of a counterexample, or otherwise, that not all cubic equations of the form

$$x^3 + \alpha x^2 + \beta x + \gamma = 0$$

can be solved by this method.

2 Let

$$\tan x = \sum_{n=0}^{\infty} a_n x^n \quad \text{for small } x,$$

$$x\cot x = 1 + \sum_{n=1}^{\infty} b_n x^n \quad \text{for small } x \text{ and not zero.}$$

Using the relation

$$\cot x - \tan x = 2\cot 2x, \qquad (*)$$

or otherwise, prove that $a_{n-1} = (1 - 2^n)b_n$, for $n \geqslant 1$.

Let

$$x\mathrm{cosec}x = 1 + \sum_{n=1}^{\infty} c_n x^n \quad \text{for small } x \neq 0.$$

Using a relation similar to $(*)$ involving $2\mathrm{cosec}2x$, or otherwise, prove that

$$c_n = \frac{2^{n-1} - 1}{2^n - 1}\frac{1}{2^{n-1}}a_{n-1} \qquad (n \geqslant 1).$$

3 The real numbers x and y are related to the real numbers u and v by

$$2(u + \mathrm{i}v) = \mathrm{e}^{x+\mathrm{i}y} - \mathrm{e}^{-x-\mathrm{i}y}.$$

Show that the line in the x-y plane given by $x = a$, where a is a positive constant, corresponds to the ellipse

$$\left(\frac{u}{\sinh a}\right)^2 + \left(\frac{v}{\cosh a}\right)^2 = 1$$

in the u-v plane. Show also that the line given by $y = b$, where b is a constant and $0 < \sin b < 1$, corresponds to one branch of a hyperbola in the u-v plane. Write down the u and v coordinates of one point of intersection of the ellipse and hyperbola branch, and show that the curves intersect at right-angles at this point.

Make a sketch of the u-v plane showing the ellipse, the hyperbola branch and the line segments corresponding to:

(i) $x = 0$;

(ii) $y = \frac{1}{2}\pi, \quad 0 \leqslant x \leqslant a.$

4 The function f is defined by

$$\mathrm{f}(x) = \frac{(x-a)\,(x-b)}{(x-c)\,(x-d)} \qquad (x \neq c,\ x \neq d)\,,$$

where a, b, c and d are real and distinct, and $a + d \neq c + d$. Show that

$$\frac{x\mathrm{f}'(x)}{\mathrm{f}(x)} = \left(1 - \frac{a}{x}\right)^{-1} + \left(1 - \frac{b}{x}\right)^{-1} - \left(1 - \frac{c}{x}\right)^{-1} - \left(1 - \frac{d}{x}\right)^{-1},$$

$(x \neq 0, x \neq a, x \neq b)$ and deduce that when $|x|$ is much larger than each of $|a|\,, |b|\,, |c|$ and $|d|\,$, the gradient of $\mathrm{f}(x)$ has the same sign as $(a + b - c - d)$.

It is given that there is a real value of real value of x for which $\mathrm{f}(x)$ takes the real value z if and only if

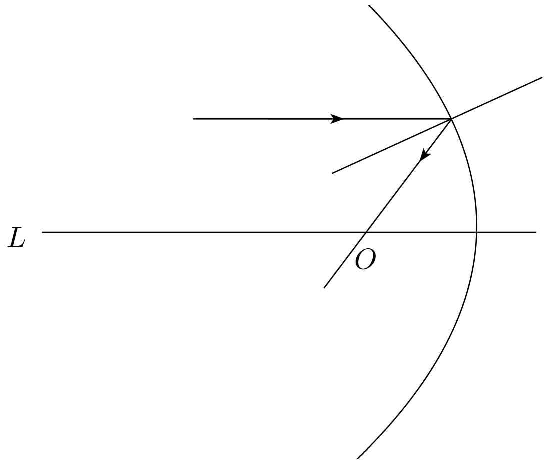
$$\left[(c-d)^2 z + (a-c)\,(b-d) + (a-d)\,(b-c)\right]^2 \geqslant 4\,(a-c)\,(b-d)\,(a-d)\,(b-c)\,.$$

Describe briefly a method by which this result could be proved, but do not attempt to prove it.

Given that $a < b$ and $a < c < d$, make sketches of the graph of f in the four distinct cases which arise, indicating the cases for which the range of f is not the whole of $\mathbb{R}$.

5 **(i)** Show that in polar coordinates, the gradient of any curve at the point (r, θ) is

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta + r\right) \Big/ \left(\frac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta\right).$$



(ii) A mirror is designed so that any ray of light which hits one side of the mirror and which is parallel to a certain fixed line L is reflected through a fixed point O on L. For any ray hitting the mirror, the normal to the mirror at the point of reflection bisects the angle between the incident ray and the reflected ray, as shown in the figure. Prove that the mirror intersects any plane containing L in a parabola.

6 The function f satisfies the condition $\mathrm{f}'(x) > 0$ for $a \leqslant x \leqslant b$, and g is the inverse of f. By making a suitable change of variable, prove that

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = b\beta - a\alpha - \int_\alpha^\beta \mathrm{g}(y)\,\mathrm{d}y,$$

where $\alpha = \mathrm{f}(a)$ and $\beta = \mathrm{f}(b)$. Interpret this formula geometrically, in the case where α and a are both positive.

Prove similarly and interpret (for $\alpha > 0$ and $a > 0$) the formula

$$2\pi \int_a^b x\mathrm{f}(x)\,\mathrm{d}x = \pi(b^2\beta - a^2\alpha) - \pi \int_\alpha^\beta [\mathrm{g}(y)]^2\,\mathrm{d}y.$$

7 By means of the substitution x^α, where α is a suitably chosen constant, find the general solution for $x > 0$ of the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - b\frac{\mathrm{d}y}{\mathrm{d}x} + x^{2b+1}y = 0,$$

where b is a constant and $b > -1$.

Show that, if $b > 0$, there exist solutions which satisfy $y \to 1$ and $\mathrm{d}y/\mathrm{d}x \to 0$ as $x \to 0$, but that these conditions do not determine a unique solution. For what values of b do these conditions determine a unique solution?

8 Let $\Omega = \exp(\mathrm{i}\pi/3)$. Prove that $\Omega^2 - \Omega + 1 = 0$.

Two transformations, R and T, of the complex plane are defined by

$$R : z \longmapsto \Omega^2 z \qquad \text{and} \qquad T : z \longmapsto \frac{\Omega z + \Omega^2}{2\Omega^2 z + 1}.$$

Verify that each of R and T permute the four point $z_0 = 0,\ z_1 = 1,\ z_2 = \Omega^2$ and $z_3 = -\Omega$. Explain, without explicitly producing a group multiplication table, why the smallest group of transformations which contains elements R and T has order at least 12.

Are there any permutations of these points which cannot be produced by repeated combinations of R and T?

9 The matrix $\mathbf{F}$ is defined by

$$\mathbf{F} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \mathbf{A}^n,$$

where $\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 8 & 3 \end{pmatrix}$, and t is a variable scalar. Evaluate $\mathbf{A}^2$, and show that

$$\mathbf{F} = \mathbf{I}\cosh t + \mathbf{A}\sinh t.$$

Show also that $\mathbf{F}^{-1} = \mathbf{I}\cosh t - \mathbf{A}\sinh t$, and that $\dfrac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \mathbf{F}\mathbf{A}$.

The vector $\mathbf{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ satisfies the differential equation

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \mathbf{A}\mathbf{r} = \mathbf{0},$$

with $x = \alpha$ and $y = \beta$ at $t = 0$. Solve this equation by means of a suitable matrix integrating factor, and hence show that

$$\begin{aligned} x(t) &= \alpha\cosh t + (3\alpha + \beta)\sinh t \\ y(t) &= \beta\cosh t - (8\alpha + 3\beta)\sinh t. \end{aligned}$$

10 State carefully the conditions which the fixed vectors $\mathbf{a}, \mathbf{b}, \mathbf{u}$ and $\mathbf{v}$ must satisfy in order to ensure that the line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$ intersects the line $\mathbf{r} = \mathbf{b} + \mu \mathbf{v}$ in exactly one point.

Find the two values of the fixed scalar b for which the planes with equations

$$\left.\begin{aligned} x + y + bz &= b + 2 \\ bx + by + z &= 2b + 1 \end{aligned}\right\} \qquad (*)$$

do not intersect in a line. For other values of b, express the line of intersection of the two planes in the form $\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$, where $\mathbf{a} \cdot \mathbf{u} = 0$.

Find the conditions which b and the fixed scalars c and d must satisfy to ensure that there is exactly one point on the line

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} + \mu \begin{pmatrix} 1 \\ d \\ 0 \end{pmatrix}$$

whose coordinates satisfy both equations $(*)$.

Section B: Mechanics

11 A lift of mass M and its counterweight of mass M are connected by a light inextensible cable which passes over a light frictionless pulley. The lift is constrained to move vertically between smooth guides. The distance between the floor and the ceiling of the lift is h. Initially, the lift is at rest, and the distance between the top of the lift and the pulley is greater than h. A small tile of mass m becomes detached from the ceiling of the lift. Show that the time taken for it to fall to the floor is

$$t = \sqrt{\frac{2\,(M-m)\,h}{Mg}}.$$

The collision between the tile and the lift floor is perfectly inelastic. Show that the lift is reduced to rest by the collision, and that the loss of energy of the system is mgh.

12 A uniform rectangular lamina of sides $2a$ and $2b$ rests in a vertical plane. It is supported in equilibrium by two smooth pegs fixed in the same horizontal plane, a distance d apart, so that one corner of the lamina is below the level of the pegs. Show that if the distance between this (lowest) corner and the peg upon which the side of length $2a$ rests is less than a, then the distance between this corner and the other peg is less than b.

Show also that

$$b\cos\theta - a\sin\theta = d\cos 2\theta,$$

where θ is the acute angle which the sides of length $2b$ make with the horizontal.

13 A body of mass m and centre of mass O is said to be *dynamically equivalent* to a system of particles of total mass m and centre of mass O if the moment of inertia of the system of particles is the same as the moment of inertia of the body, about any axis through O. Show that this implies that the moment of inertia of the system of particles is the same as that of the body about *any* axis.

Show that a uniform rod of length $2a$ and mass m is dynamically equivalent to a suitable system of three particles, one at each end of the rod, and one at the midpoint.

Use this result to deduce that a uniform rectangular lamina of mass M is dynamically equivalent to a system consisting of particles each of mass $\frac{1}{36}M$ at the corners, particles each of mass $\frac{1}{9}M$ at the midpoint of each side, and a particle of mass $\frac{4}{9}M$ at the centre. Hence find the moment of inertia of a square lamina, of side $2a$ and mass M, about one of its diagonals.

The mass per unit length of a thin rod of mass m is proportional to the distance from one end of the rod, and a dynamically equivalent system consists of one particle at each end of the rod and one at the midpoint. Write down a set of equations which determines these masses, and show that, in fact, only two particles are required.

14 One end of a light inextrnsible string of length l is fixed to a point on the upper surface of a thin, smooth, horizontal table-top, at a distance $(l - a)$ from one edge of the table-top. A particle of mass m is fixed to the other end of the string, and held a distance a away from this edge of the table-top, so that the string is horizontal and taut. The particle is then released. Find the tension in the string after the string has rotated through an angle θ, and show that the largest magnitude of the force on the edge of the table top is $8mg/\sqrt{3}$.

Section C: Probability and Statistics

15 Two points are chosen independently at random on the perimeter (including the diameter) of a semicircle of unit radius. What is the probability that exactly one of them lies on the diameter?

Let the area of the triangle formed by the two points and the midpoint of the diameter be denoted by the random variable A.

(i) Given that exactly one point lies on the diameter, show that the expected value of A is $(2\pi)^{-1}$.

(ii) Given that neither point lies on the diameter, show that the expected value of A is π^{-1}. [You may assume that if two points are chosen at random on a line of length π units, the probability density function for the distance X between the two points is $2\left(\pi - x\right)/\pi^2$ for $0 \leqslant x \leqslant \pi$.]

Using these results, or otherwise, show that the expected value of A is $(2+\pi)^{-1}$.

16 Widgets are manufactured in batches of size $(n+N)$. Any widget has a probability p of being faulty, independent of faults in other widgets. The batches go through a quality control procedure in which a sample of size n, where $n \geqslant 2$, is taken from each batch and tested. If two or more widgets in the sample are found to be faulty, all widgets in the batch are tested and all faults corrected. If fewer than two widgets in the sample are found to be faulty, the sample is replaced in the batch and no faults are corrected. Show that the probability that the batch contains exactly k, where $k \leqslant N$, faulty widgets after quality control is

$$\frac{\left[N+1+k\left(n-1\right)\right]N!}{(N-k+1)!k!}p^k\left(1-p\right)^{N+n-k},$$

and verify that this formula also gives the correct answer for $k = N+1$.

Show that the expected number of faulty widgets in a batch after quality control is

$$\left[N+n+pN(n-1)\right]p(1-p)^{n-1}.$$