Section A: Pure Mathematics

1 Prove that both $x^4 - 2x^3 + x^2$ and $x^2 - 8x + 17$ are non-negative for all real x. By considering the intervals $x \leqslant 0$, $0 < x \leqslant 2$ and $x > 2$ separately, or otherwise, prove that the equation

$$x^4 - 2x^3 + x^2 - 8x + 17 = 0$$

has no real roots.

Prove that the equation $x^4 - x^3 + x^2 - 4x + 4 = 0$ has no real roots.

2 Prove that if $A + B + C + D = \pi$, then

$$\sin(A+B)\sin(A+D) - \sin B \sin D = \sin A \sin C.$$

The points P, Q, R and S lie, in that order, on a circle of centre O. Prove that

$$PQ \times RS + QR \times PS = PR \times QS.$$

3 Sketch the curves given by

$$y = x^3 - 2bx^2 + c^2x,$$

where b and c are non-negative, in the cases:

$$\begin{array}{l}(\mathbf{i})\ \ 2b < c\sqrt{3}, \quad (\mathbf{ii})\ \ 2b = c\sqrt{3} \neq 0, \quad (\mathbf{iii})\ \ c\sqrt{3} < 2b < 2c, \quad (\mathbf{iv})\ \ b = c \neq 0, \\ (\mathbf{v})\ \ b > c > 0, \quad (\mathbf{vi})\quad c = 0, b \neq 0, \quad (\mathbf{vii})\ \ c = b = 0.\end{array}$$

Sketch also the curves given by $y^2 = x^3 - 2bx^2 + c^2x$ in the cases **(i)**, **(v)** and **(vii)**.

4 A plane contains n distinct given lines, no two of which are parallel, and no three of which intersect at a point. By first considering the cases $n = 1, 2, 3$ and 4, provide and justify, by induction or otherwise, a formula for the number of line segments (including the infinite segments).

Prove also that the plane is divided into $\frac{1}{2}(n^2 + n + 2)$ regions (including those extending to infinity).

5 The distinct points L, M, P and Q of the Argand diagram lie on a circle S centred on the origin and the corresponding complex numbers are l, m, p and q. By considering the perpendicular bisectors of the chords, or otherwise, prove that the chord LM is perpendicular to the chord PQ if and only if $lm + pq = 0$.

Let A_1, A_2 and A_3 be three distinct points on S. For any given point A_1' on S, the points A_2', A_3' and A_1'' are chosen on S such that $A_1'A_2', A_2'A_3'$ and $A_3'A_1''$ are perpendicular to A_1A_2, A_2A_3 and $A_3A_1,$ respectively. Show that for exactly two positions of $A_1',$ the points A_1' and A_1'' coincide.

If, instead, A_1, A_2, A_3 and A_4 are four given distinct points on S and, for any given point A_1', the points A_2', A_3', A_4' and A_1'' are chosen on S such that $A_1'A_2', A_2'A_3', A_3'A_4'$ and $A_4'A_1''$ are respectively perpendicular to A_1A_2, A_2A_3, A_3A_4 and $A_4A_1,$ show that A_1' coincides with A_1''.

Give the corresponding result for n distinct points on S.

6 Let a, b, c, d, p and q be positive integers. Prove that:

(i) if $b > a$ and $c > 1,$ then $bc \geqslant 2c \geqslant 2 + c$;

(ii) if $a < b$ and $d < c$, then $bc - ad \geqslant a + c$;

(iii) if $\dfrac{a}{b} < p < \dfrac{c}{d}$, then $(bc - ad)\,p \geqslant a + c$;

(iv) if $\dfrac{a}{b} < \dfrac{p}{q} < \dfrac{c}{d}$, then $p \geqslant \dfrac{a+c}{bc-ad}$ and $q \geqslant \dfrac{b+d}{bc-ad}$.

Hence find all fractions with denominators less than 20 which lie between $8/9$ and $9/10$.

7 A damped system with feedback is modelled by the equation

$$\mathrm{f}'(t) + \mathrm{f}(t) - k\mathrm{f}(t-1) = 0, \qquad (\dagger)$$

where k is a given non-zero constant. Show that (non-zero) solutions for f of the form $\mathrm{f}(t) = A\mathrm{e}^{pt}$, where A and p are constants, are possible provided p satisfies

$$p + 1 = k\mathrm{e}^{-p}. \qquad (*)$$

Show also, by means of a sketch, or otherwise, that equation $(*)$ can have $0, 1$ or 2 real roots, depending on the value of k, and find the set of values of k for which such solutions of $(\dagger)$ exist. For what set of values of k do such solutions tend to zero as $t \to +\infty$?

8 The functions x and y are related by

$$\mathrm{x}(t) = \int_0^t \mathrm{y}(u)\,\mathrm{d}u,$$

so that $\mathrm{x}'(t) = \mathrm{y}(t)$. Show that

$$\int_0^1 \mathrm{x}(t)\mathrm{y}(t)\,\mathrm{d}t = \tfrac{1}{2}\left[\mathrm{x}(1)\right]^2.$$

In addition, it is given that $\mathrm{y}(t)$ satisfies

$$\mathrm{y}'' + (\mathrm{y}^2 - 1)\mathrm{y}' + \mathrm{y} = 0, \ (*)$$

with $\mathrm{y}(0) = \mathrm{y}(1)$ and $\mathrm{y}'(0) = \mathrm{y}'(1)$. By integrating $(*)$, prove that $\mathrm{x}(1) = 0$.

By multiplying $(*)$ by $\mathrm{x}(t)$ and integrating by parts, prove the relation

$$\int_0^1 \left[\mathrm{y}(t)\right]^2\,\mathrm{d}t = \tfrac{1}{3}\int_0^1 \left[\mathrm{y}(t)\right]^4\,\mathrm{d}t.$$

Prove also the relation

$$\int_0^1 \left[\mathrm{y}'(t)\right]^2\,\mathrm{d}t = \int_0^1 \left[\mathrm{y}(t)\right]^2\,\mathrm{d}t.$$

9 Show by means of a sketch that the parabola $r(1+\cos\theta) = 1$ cuts the interior of the cardioid $r = 4(1+\cos\theta)$ into two parts.

Show that the total length of the boundary of the part that includes the point $r = 1, \theta = 0$ is $18\sqrt{3} + \ln(2+\sqrt{3})$.

10 Two square matrices $\mathbf{A}$ and $\mathbf{B}$ satisfies $\mathbf{AB} = \mathbf{0}$. Show that either $\det\mathbf{A} = 0$ or $\det\mathbf{B} = 0$ or $\det\mathbf{A} = \det\mathbf{B} = 0$. If $\det\mathbf{B} \neq 0$, what must $\mathbf{A}$ be? Give an example to show that the condition $\det\mathbf{A} = \det\mathbf{B} = 0$ is not sufficient for the equation $\mathbf{AB} = \mathbf{0}$ to hold.

Find real numbers p, q and r such that

$$\mathbf{M}^3 + 2\mathbf{M}^2 - 5\mathbf{M} - 6\mathbf{I} = (\mathbf{M} + p\mathbf{I})(\mathbf{M} + q\mathbf{I})(\mathbf{M} + r\mathbf{I}),$$

where $\mathbf{M}$ is any square matrix and $\mathbf{I}$ is the appropriate identity matrix.

Hence, or otherwise, find all matrices $\mathbf{M}$ of the form $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ which satisfy the equation

$$\mathbf{M}^3 + 2\mathbf{M}^2 - 5\mathbf{M} - 6\mathbf{I} = \mathbf{0}.$$

Section B: Mechanics

11 A disc is free to rotate in a horizontal plane about a vertical axis through its centre. The moment of inertia of the disc about this axis is mk^2. Along one diameter is a narrow groove in which a particle of mass m slides freely. At time $t = 0$, the disc is rotating with angular speed Ω, and the particle is at a distance a from the axis and is moving towards the axis with speed V, where $k^2V^2 = \Omega^2 a^2(k^2 + a^2)$. Show that, at a later time t, while the particle is still moving towards the axis, the angular speed ω of the disc and the distance r of the particle from the axis are related by

$$\omega = \frac{\Omega(k^2 + a^2)}{k^2 + r^2} \qquad \text{and} \qquad \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\Omega r(k^2 + a^2)}{k(k^2 + r^2)^{\frac{1}{2}}}.$$

Deduce that

$$k\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r(k^2 + r^2)^{\frac{1}{2}},$$

where θ is the angle through which the disc has turned at time t. By making the substitution $u = 1/r$, or otherwise, show that $r\sinh(\theta + \alpha) = k$, where $\sinh\alpha = k/a$. Hence, or otherwise, show that the particle never reaches the axis.

12 A straight staircase consists of N smooth horizontal stairs each of height h. A particle slides over the top stair at speed U, with velocity perpendicular to the edge of the stair, and then falls down the staircase, bouncing once on every stair. The coefficient of restitution between the particle and each stair is e, where $e < 1$. Show that the horizontal distance d_n travelled between the nth and $(n+1)$th bounces is given by

$$d_n = U\left(\frac{2h}{g}\right)^{\frac{1}{2}}\left(e\alpha_n + \alpha_{n+1}\right),$$

where $\alpha_n = \left(\dfrac{1 - e^{2n}}{1 - e^2}\right)^{\frac{1}{2}}$.

If N is very large, show that U must satisfy

$$U = \left(\frac{L^2 g}{2h}\right)^{\frac{1}{2}}\left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}},$$

where L is the horizontal distance between the edges of successive stairs.

13 A thin non-uniform rod PQ of length $2a$ has its centre of gravity a distance $a + d$ from P. It hangs (not vertically) in equilibrium suspended from a small smooth peg O by means of a light inextensible string of length $2b$ which passes over the peg and is attached at its ends to P and Q. Express OP and OQ in terms of a, b and d. By considering the angle POQ, or otherwise, show that $d < a^2/b$.

14 The identical uniform smooth spherical marbles $A_1, A_2, \dots, A_n$, where $n \geqslant 3$, each of mass m, lie in that order in a smooth straight trough, with each marble touching the next. The marble A_{n+1}, which is similar to A_n but has mass λm, is placed in the trough so that it touches A_n. Another marble A_0, identical to A_n, slides along the trough with speed u and hits A_1. It is given that kinetic energy is conserved throughout.

(i) Show that if $\lambda < 1$, there is a possible subsequent motion in which only A_n and A_{n+1} move (and A_0 is reduced to rest), but that if $\lambda > 1$, such a motion is not possible.

(ii) If $\lambda > 1$, show that a subsequent motion in which only A_{n-1}, A_n and A_{n+1} move is not possible.

(iii) If $\lambda > 1$, find a possible subsequent motion in which only two marbles move.

Section C: Probability and Statistics

15 A target consists of a disc of unit radius and centre O. A certain marksman never misses the target, and the probability of any given shot hitting the target within a distance t from O it t^2, where $0 \leqslant t \leqslant 1$. The marksman fires n shots independently. The random variable Y is the radius of the smallest circle, with centre O, which encloses all the shots. Show that the probability density function of Y is $2ny^{2n-1}$ and find the expected area of the circle.

The shot which is furthest from O is rejected. Show that the expected area of the smallest circle, with centre O, which encloses the remaining $(n-1)$ shots is

$$\left(\frac{n-1}{n+1}\right)\pi.$$

16 Each day, I choose at random between my brown trousers, my grey trousers and my expensive but fashionable designer jeans. Also in my wardrobe, I have a black silk tie, a rather smart brown and fawn polka-dot tie, my regimental tie, and an elegant powder-blue cravat which I was given for Christmas. With my brown or grey trousers, I choose ties (including the cravat) at random, except of course that I don't wear the cravat with the brown trousers or the polka-dot tie with the grey trousers. With the jeans, the choice depends on whether it is Sunday or one of the six weekdays: on weekdays, half the time I wear a cream-coloured sweat-shirt with $E = mc^2$ on the front and no tie; otherwise, and on Sundays (when naturally I always wear a tie), I just pick at random from my four ties.

This morning, I received through the post a compromising photograph of myself. I often receive such photographs and they are equally likely to have been taken on any day of the week. However, in this particular photograph, I am wearing my black silk tie. Show that, on the basis of this information, the probability that the photograph was taken on Sunday is $11/68$.

I should have mentioned that on Mondays I lecture on calculus and I therefore always wear my jeans (to make the lectures seem easier to understand). Find, on the basis of the complete information, the probability that the photograph was taken on Sunday.

[The phrase 'at random' means 'with equal probability'.]