Section A: Pure Mathematics

1 If $\theta + \phi + \psi = \frac{1}{2}\pi$, show that

$$\sin^2\theta + \sin^2\phi + \sin^2\psi + 2\sin\theta\sin\phi\sin\psi = 1.$$

By taking $\theta = \phi = \frac{1}{5}\pi$ in this equation, or otherwise, show that $\sin\frac{1}{10}\pi$ satisfies the equation

$$8x^3 + 8x^2 - 1 = 0.$$

2 Frosty the snowman is made from two uniform spherical snowballs, of initial radii $2R$ and $3R$. The smaller (which is his head) stands on top of the larger. As each snowball melts, its volume decreases at a rate which is directly proportional to its surface area, the constant of proportionality being the same for both snowballs. During melting each snowball remains spherical and uniform. When Frosty is half his initial height, find the ratio of his volume to his initial volume.

If V and S denote his total volume and surface area respectively, find the maximum value of $\dfrac{\mathrm{d}V}{\mathrm{d}S}$ up to the moment when his head disappears.

3 A path is made up in the Argand diagram of a series of straight line segments P_1P_2, P_2P_3, $P_3P_4, \ldots$ such that each segment is d times as long as the previous one, $(d \neq 1)$, and the angle between one segment and the next is always θ (where the segments are directed from P_j towards P_{j+1}, and all angles are measured in the anticlockwise direction). If P_j represents the complex number z_j, express

$$\frac{z_{n+1} - z_n}{z_n - z_{n-1}}$$

as a complex number (for each $n \geqslant 2$), briefly justifying your answer.

If $z_1 = 0$ and $z_2 = 1$, obtain an expression for z_{n+1} when $n \geqslant 2$. By considering its imaginary part, or otherwise, show that if $\theta = \frac{1}{3}\pi$ and $d = 2$, then the path crosses the real axis infinitely often.

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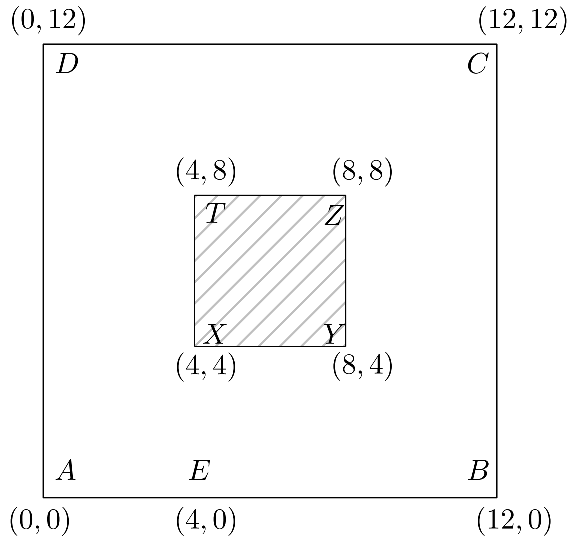

The above diagram is a plan of a prison compound. The outer square $ABCD$ represents the walls of the compound (whose height may be neglected), while the inner square $XYZT$ is the Black Tower, a solid stone structure. A guard patrols along segment AE of the walls, for a distance of up to 4 units from A. Determine the distance from A of points at which the area of the courtyard that he can see is

(i) as small as possible,

(ii) as large as possible.

[**Hint.** It is suggested that you express the area he *cannot* see in terms of p, his distance from A.]

5 A set of n distinct vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n,$ where $n \geqslant 2,$ is called *regular* if it satisfies the following two conditions:

(i) there are constants α and β, with $\alpha > 0$, such that for any i and j,

$$\mathbf{a}_i \cdot \mathbf{a}_j = \begin{cases} \alpha^2 & \text{when } i = j \\ \beta & \text{when } i \neq j, \end{cases}$$

(ii) the centroid of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is the origin $\mathbf{0}$. [The centroid of vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m$ is the vector $\frac{1}{m}(\mathbf{b}_1 + \mathbf{b}_2 + \cdots + \mathbf{b}_m)$.]

Prove that **(i)** and **(ii)** imply that $(n-1)\beta = -\alpha^2$.

If $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is a regular set of vectors in 2-dimensional space, show that either $n = 2$ or $n = 3$, and in each case find the other vectors in the set.

Hence, or otherwise, find all regular sets of vectors in 3-dimensional space for which $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{a}_2$ lies in the x-y plane.

6 Criticise each step of the following arguments. You should correct the arguments where necessary and possible, and say (with justification) whether you think the conclusion are true even though the argument is incorrect.

(i) The function g defined by

$$\mathrm{g}(x) = \frac{2x^3 + 3}{x^4 + 4}$$

satisfies $\mathrm{g}'(x) = 0$ only for $x = 0$ or $x = \pm 1$. Hence the stationary values are given by $x = 0$, $\mathrm{g}(x) = \frac{3}{4}$ and $x = \pm 1$, $\mathrm{g}(x) = 1$. Since $\frac{3}{4} < 1$, there is a minimum at $x = 0$ and maxima at $x = \pm 1$. Thus we must have $\frac{3}{4} \leqslant \mathrm{g}(x) \leqslant 1$ for all x.

(ii) $\displaystyle\int (1-x)^{-3}\,\mathrm{d}x = -3(1-x)^{-4}$ and so $\displaystyle\int_{-1}^{3} (1-x)^{-3}\,\mathrm{d}x = 0.$

7 According to the Institute of Economic Modelling Sciences, the Slakan economy has alternate years of growth and decline, as in the following model. The number V of vloskan (the unit of currency) in the Slakan Treasury is assumed to behave as a continuous variable, as follows. In a year of growth it increases continuously at an annual rate $aV_0\left(1+(V/V_0)\right)^2$. During a year of decline, as long as there is still money in the Treasury, the amount decreases continuously at an annual rate $bV_0\left(1+(V/V_0)\right)^2$; but if V becomes zero, it remains zero until the end of the year. Here a, b and V_0 are positive constants. A year of growth has just begun and there are k_0V_0 vloskan in the Treasury, where $0 \leqslant k_0 < a^{-1} - 1$. Explain the significance of these inequalities for the model to be remotely sensible.

If k_0 is as above and at the end of one year there are k_1V_0 vloskan in the Treasury, where $k_1 > 0$, find the condition involving b which k_1 must satisfy so that there will be some vloskan left after a further year. Under what condition (involving a, b and k_0) does the model predict that unlimited growth will take place in the third year (but not before)?

8 **(i)** By a substitution of the form $y = k - x$ for suitable k, prove that, for any function f,

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x = \pi \int_0^{\frac{1}{2}\pi} \mathrm{f}(\sin x)\,\mathrm{d}x.$$

Hence or otherwise evaluate

$$\int_0^{\pi} \frac{x}{2+\sin x}\,\mathrm{d}x.$$

(ii) Evaluate

$$\int_0^1 \frac{(\sin^{-1} t)\cos\left[(\sin^{-1} t)^2\right]}{\sqrt{1-t^2}}\,\mathrm{d}t.$$

[No credit will be given for numerical answers obtained by use of a calculator.]

9 **(i)** Suppose that the real number x satisfies the n inequalities

$$\begin{aligned} 1 &< x \ \ < 2 \\ 2 &< x^2 < 3 \\ 3 &< x^3 < 4 \\ &\ \ \vdots \\ n &< x^n < n+1 \end{aligned}$$

Prove without the use of a calculator that $n \leqslant 4$.

(ii) If n is an integer strictly greater than 1, by considering how many terms there are in

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n^2},$$

or otherwise, show that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n^2} > 1.$$

Hence or otherwise find, with justification, an integer N such that $\displaystyle\sum_{n=1}^{N} \frac{1}{n} > 10$.

Section B: Mechanics

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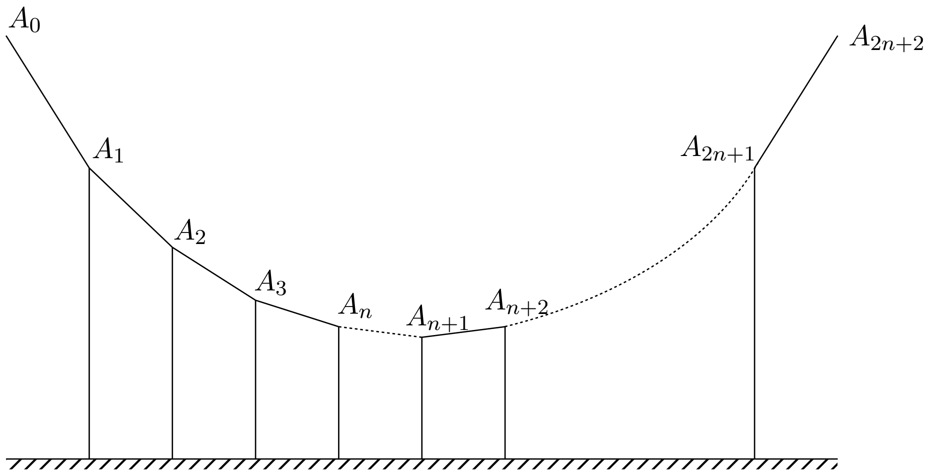

The above diagram represents a suspension bridge. A heavy uniform horizontal roadway is attached by vertical struts to a light flexible chain at points $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2), \ldots,$ $A_{2n+1} = (x_{2n+1}, y_{2n+1})$, where the coordinates are referred to horizontal and vertically upward axes Ox, Oy. The chain is fixed to external supports at points

$$A_0 = (x_0, y_0) \quad \text{and} \quad A_{2n+2} = (x_{2n+2}, y_{2n+2})$$

at the same height. The weight of the chain and struts may be neglected. Each strut carries the same weight w. The horizontal spacing h between A_i and A_{i+1} (for $0 \leqslant i \leqslant 2n+1$) is constant. Write down equations satisfied by the tensions T_i in the portion $A_{i-1}A_i$ of the chain for $1 \leqslant i \leqslant n+1$. Hence or otherwise show that

$$\frac{h}{y_n - y_{n+1}} = \frac{3h}{y_{n-1} - y_n} = \cdots = \frac{(2n+1)y}{y_0 - y_1}.$$

Verify that the points $A_0, A_1, \ldots, A_{2n+1}, A_{2n+2}$ lie on a parabola.

11 A piledriver consists of a weight of mass M connected to a lighter counterweight of mass m by a light inextensible string passing over a smooth light fixed pulley. By considerations of energy or otherwise, show that if the weights are released from rest, and move vertically, then as long as the string remains taut and no collisions occur, the weights experience a constant acceleration of magnitude

$$g\left(\frac{M-m}{M+m}\right).$$

Initially the weight is held vertically above the pile, and is released from rest. During the subsequent motion both weights move vertically and the only collisions are between the weight and the pile. Treating the pile as fixed and the collisions as completely inelastic, show that, if just before a collision the counterweight is moving with speed v, then just before the next collision it will be moving with speed $mv/(M+m)$. [You may assume that when the string becomes taut, the momentum lost by one weight equals that gained by the other.]

Further show that the times between successive collisions with the pile form a geometric progression. Show that the total time before the weight finally comes to rest is three times the time from the start to the first impact.

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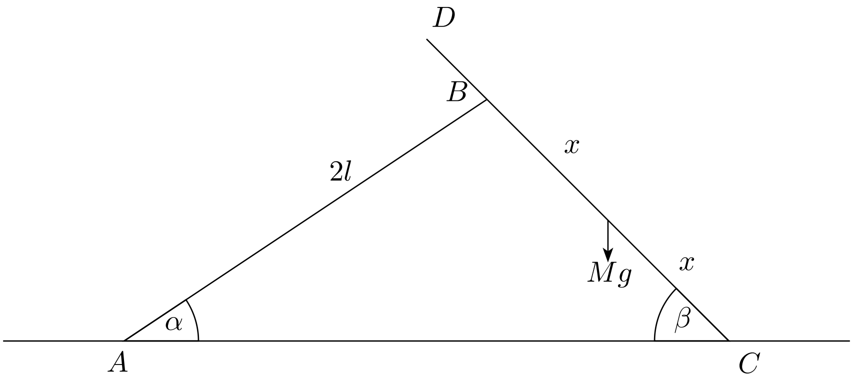

The above diagram illustrates a makeshift stepladder, made from two equal light planks AB and CD, each of length $2l$. The plank AB is smoothly hinged to the ground at A and makes an angle of α with the horizontal. The other plank CD has its bottom end C resting on the same horizontal ground and makes an angle β with the horizontal. It is pivoted smoothly to B at a point distance $2x$ from C. The coefficient of friction between CD and the ground is μ. A painter of mass M stands on CD, half between C and B. Show that, for equilibrium to be possible,

$$\mu \geqslant \frac{\cot\alpha\cot\beta}{2\cot\alpha+\cot\beta}.$$

Suppose now that B coincides with D. Show that, as α varies, the maximum distance from A at which the painter will be standing is

$$l\sqrt{\frac{1+81\mu^2}{1+9\mu^2}}.$$

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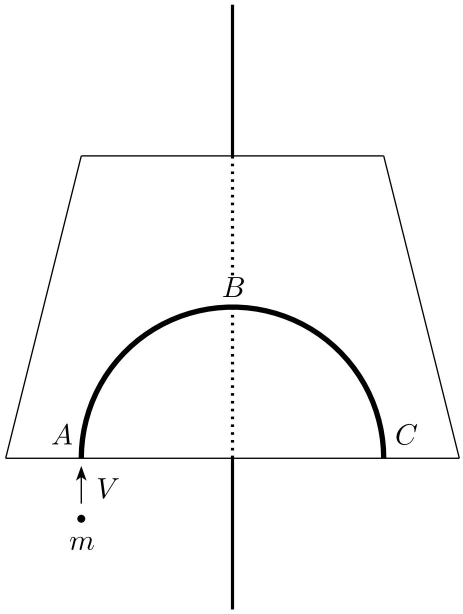

A heavy smooth lamina of mass M is free to slide without rotation along a straight line on a fixed smooth horizontal table. A smooth groove ABC is inscribed in the lamina, as indicated in the above diagram. The tangents to the groove at A and at B are parallel to the line. When the lamina is stationary, a particle of mass m (where $m < M$) enters the groove at A. The particle is travelling, with speed V, parallel to the line and in the plane of the lamina and table. Calculate the speeds of the particle and of the lamina, when the particle leaves the groove at C.

Suppose now that the lamina is held fixed by a peg attached to the line. Supposing that the groove ABC is a semicircle of radius r, obtain the value of the average force per unit time exerted on the peg by the lamina between the instant that the particle enters the groove and the instant that it leaves it.

Section C: Probability and Statistics

14 A set of $2N+1$ rods consists of one of each length $1, 2, \ldots, 2N, 2N+1$, where N is an integer greater than 1. Three different rods are selected from the set. Suppose their lengths are a, b and c, where $a > b > c$. Given that a is even and fixed, show, by considering the possible values of b, that the number of selections in which a triangle can then be formed from the three rods is

$$1 + 3 + 5 + \cdots + (a-3),$$

where we allow only non-degenerate triangles (i.e. triangles with non-zero area).

Similarly obtain the number of selections in which a triangle may be formed when a takes some fixed odd value. Write down a formula for the number of ways of forming a non-degenerate triangle and verify it for $N = 3$.

Hence show that, if three rods are drawn at random without replacement, then the probability that they can form a non-degenerate triangle is

$$\frac{(N-1)(4N+1)}{2(4N^2-1)}.$$

15 A fair coin is thrown n times. On each throw, 1 point is scored for a head and 1 point is lost for a tail. Let S_n be the points total for the series of n throws, i.e. $S_n = X_1 + X_2 + \cdots + X_n$, where

$$X_j = \begin{cases} 1 & \text{if the } j\text{th throw is a head} \\ -1 & \text{if the } j\text{th throw is a tail.} \end{cases}$$

(i) If $n = 10\,000$, find an approximate value for the probability that $S_n > 100$.

(ii) Find an approximate value for the least n for which $\mathrm{P}(S_n > 0.01n) < 0,01$.

Suppose that instead no points are scored for the first throw, but that on each successive threw, 2 points are scored if both it and the first throw are heads, two points are deducted if both are tails, and no points are scored or lost if the throws differ. Let Y_k be the score on the kth throw, where $2 \leqslant k \leqslant n$. Show that $Y_k = X_1 + X_k$.

Calculate the mean and variance of each Y_k and determine whether it is true that

$$\mathrm{P}(Y_2 + Y_3 + \cdots + Y_n > 0.01(n-1)) \to 0 \quad \text{as } n \to \infty.$$

16 At any instant the probability that it is safe to cross a busy road is 0.1. A toad is waiting to cross this road. Every minute she looks at the road. If it is safe, she will cross; if it is not safe, she will wait for a minute before attempting to cross again. Find the probability that she eventually crosses the road without mishap.

Later on, a frog is also trying to cross the same road. He also inspects the traffic at one minute intervals and crosses if it is safe. Being more impatient than the toad, he may also attempt to cross when it is not safe. The probability that he will attempt to cross when it is not safe is $n/3$ if $n \leqslant 3,$ where n minutes have elapsed since he firrst inspected the road. If he attempts to cross when it is not safe, he is run over with probability $0.8,$ but otherwise he reaches the other side safely. Find the probability that he eventually crosses the road without mishap.

What is the probability that both reptiles safely cross the road with the frog taking less time than the toad? If the frog has not arrived at the other side 2 minutes after he began his attempt to cross, what is the probability that the frog is run over (at some stage) in his attempt to cross?

[Once moving, the reptiles spend a negligible time on their attempt to cross the road.]