Section A: Pure Mathematics

1 Find the limit, as $n \to \infty$, of each of the following. You should explain your reasoning briefly.

$$(\mathbf{i})\ \frac{n}{n+1}, \qquad (\mathbf{ii})\ \frac{5n+1}{n^2-3n+4}, \qquad (\mathbf{iii})\ \frac{\sin n}{n},$$

$$(\mathbf{iv})\ \frac{\sin(1/n)}{(1/n)}, \qquad (\mathbf{v})\ (\arctan n)^{-1}, \qquad (\mathbf{vi})\ \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}-\sqrt{n}}.$$

2 Suppose that y satisfies the differential equation

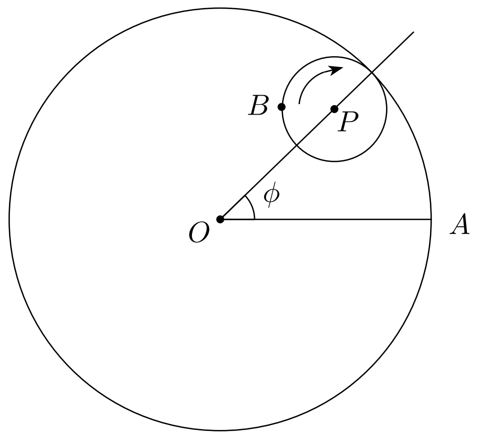
$$y = x\frac{\mathrm{d}y}{\mathrm{d}x} - \cosh\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right). \qquad (*)$$

By differentiating both sides of $(*)$ with respect to x, show that either

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \qquad \text{or} \qquad x - \sinh\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0.$$

Find the general solutions of each of these two equations. Determine the solutions of $(*)$.

3 In the figure, the large circle with centre O has radius 4 and the small circle with centre P has radius 1. The small circle rolls around the inside of the larger one. When P was on the line OA (before the small circle began to roll), the point B was in contact with the point A on the large circle.



Sketch the curve C traced by B as the circle rolls. Show that if we take O to be the origin of cartesian coordinates and the line OA to be the x-axis (so that A is the point $(4, 0)$) then B is the point

$$(3\cos\phi + \cos 3\phi, 3\sin\phi - \sin 3\phi).$$

It is given that the area of the region enclosed by the curve C is

$$\int_0^{2\pi} x \frac{\mathrm{d}y}{\mathrm{d}\phi} \,\mathrm{d}\phi,$$

where B is the point (x, y). Calculate this area.

4 $\Diamond$ is an operation which take polynomials in x to polynomials in x; that is, given a polynomial $\mathrm{h}(x)$ there is another polynomial called $\Diamond\mathrm{h}(x)$. It is given that, if $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are any two polynomials in x, the following are always true:

(i) $\Diamond(\mathrm{f}(x)\mathrm{g}(x)) = \mathrm{g}(x)\Diamond\mathrm{f}(x) + \mathrm{f}(x)\Diamond\mathrm{g}(x)$,

(ii) $\Diamond(\mathrm{f}(x) + \mathrm{g}(x)) = \Diamond\mathrm{f}(x) + \Diamond\mathrm{g}(x)$,

(iii) $\Diamond x = 1$

(iv) if λ is a constant then $\Diamond(\lambda\mathrm{f}(x)) = \lambda\Diamond\mathrm{f}(x)$.

Show that, if $\mathrm{f}(x)$ is a constant (i.e., a polynomial of degree zero), then $\Diamond\mathrm{f}(x) = 0$.

Calculate $\Diamond x^2$ and $\Diamond x^3$. Prove that $\Diamond\mathrm{h}(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{h}(x))$ for any polynomial $\mathrm{h}(x)$.

5 Explain what is meant by the order of an element g of a group G.

The set S consists of all 2×2 matrices whose determinant is 1. Find the inverse of the element $\mathbf{A}$ of S, where

$$\mathbf{A} = \begin{pmatrix} w & x \\ y & x \end{pmatrix}.$$

Show that S is a group under matrix multiplication (you may assume that matrix multiplication is associative). For which elements $\mathbf{A}$ is $\mathbf{A}^{-1} = \mathbf{A}$? Which element or elements have order 2? Show that the element $\mathbf{A}$ of S has order 3 if, and only if, $w + z + 1 = 0$. Write down one such element.

6 Sketch the graphs of $y = \sec x$ and $y = \ln(2 \sec x)$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. Show graphically that the equation

$$kx = \ln(2 \sec x)$$

has no solution with $0 \leqslant x < \frac{1}{2}\pi$ if k is a small positive number but two solutions if k is large. Explain why there is a number k_0 such that

$$k_0 x = \ln(2 \sec x)$$

has exactly one solution with $0 \leqslant x < \frac{1}{2}\pi$. Let x_0 be this solution, so that $0 \leqslant x_0 < \frac{1}{2}\pi$ and $k_0 x_0 = \ln(2 \sec x)$. Show that

$$x_0 = \cot x_0 \ln(2 \sec x_0).$$

Use any appropriate method to find x_0 correct to two decimal places. Hence find an approximate value for k_0.

7 The cubic equation

$$x^3 - px^2 + qx - r = 0$$

has roots a, b and c. Express p, q and r in terms of a, b and c.

(i) If $p = 0$ and two of the roots are equal to each other, show that

$$4q^3 + 27r^2 = 0.$$

(ii) Show that, if two of the roots of the original equation are equal to each other, then

$$4\left(q - \frac{p^2}{3}\right)^3 + 27\left(\frac{2p^3}{27} - \frac{pq}{3} + r\right)^2 = 0.$$

8 Calculate the following integrals

(i) $$\int \frac{x}{(x-1)(x^2-1)}\,\mathrm{d}x;$$

(ii) $$\int \frac{1}{3\cos x + 4\sin x}\,\mathrm{d}x;$$

(iii) $$\int \frac{1}{\sinh x}\,\mathrm{d}x.$$

9 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be the position vectors of points A, B and C in three-dimensional space. Suppose that A, B, C and the origin O are not all in the same plane. Describe the locus of the point whose position vector $\mathbf{r}$ is given by

$$\mathbf{r} = (1-\lambda-\mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where λ and μ are scalar parameters. By writing this equation in the form $\mathbf{r}\cdot\mathbf{n} = p$ for a suitable vector $\mathbf{n}$ and scalar p, show that

$$-(\lambda+\mu)\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \lambda\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) + \mu\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = 0$$

for all scalars λ, μ.

Deduce that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}).$$

Say briefly what happens if A, B, C and O are all in the same plane.

10 Let α be a fixed angle, $0 < x \leqslant \frac{1}{2}\pi$. In each of the following cases, sketch the locus of z in the Argand diagram (the complex plane):

(i) $$\arg\left(\frac{z-1}{z}\right) = \alpha,$$

(ii) $$\arg\left(\frac{z-1}{z}\right) = \alpha - \pi,$$

(iii) $$\left|\frac{z-1}{z}\right| = 1.$$

Let z_1, z_2, z_3 and z_4 be four points lying (in that order) on a circle in the Argand diagram. If

$$w = \frac{(z_1-z_2)(z_3-z_4)}{(z_4-z_1)(z_2-z_3)}$$

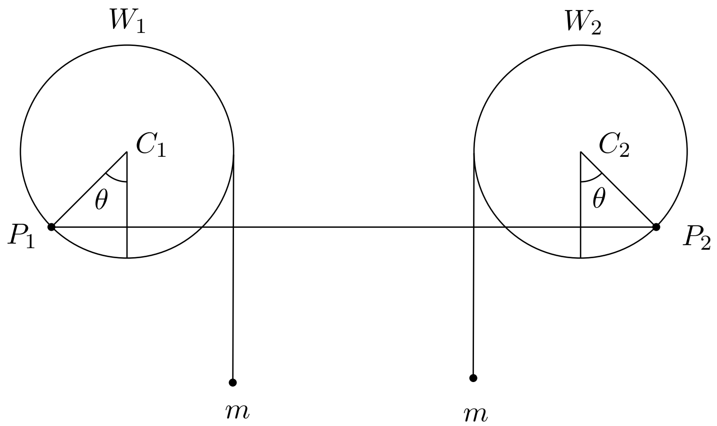
show, by considering $\arg w$, that w is real.

Section B: Mechanics

11 I am standing next to an ice-cream van at a distance d from the top of a vertical cliff of height h. It is not safe for me to go any nearer to the top of the cliff. My niece Padma is on the broad level beach at the foot of the cliff. I have just discovered that I have left my wallet with her, so I cannot buy her an ice-cream unless she can throw the wallet up to me. She can throw it at speed V, at any angle she chooses and from anywhere on the beach. Air resistance is negligible; so is Padma's height compared to that of the cliff. Show that she can throw the wallet to me if and only if

$$V^2 \geqslant g(2h + d).$$

12 In the figure, W_1 and W_2 are wheels, both of radius r. Their centres C_1 and C_2 are fixed at the same height, a distance d apart, and each wheel is free to rotate, without friction, about its centre. Both wheels are in the same vertical plane. Particles of mass m are suspended from W_1 and W_2 as shown, by light inextensible strings would round the wheels. A light elastic string of natural length d and modulus elasticity λ is fixed to the rims of the wheels at the points P_1 and P_2. The lines joining C_1 to P_1 and C_2 to P_2 both make an angle θ with the vertical. The system is in equilibrium.



Show that

$$\sin 2\theta = \frac{mgd}{\lambda r}.$$

For what value or values of λ (in terms of m, d, r and g) are there

(i) no equilibrium positions,

(ii) just one equilibrium position,

(iii) exactly two equilibrium positions,

(iv) more than two equilibrium positions?

13 Two particles P_1 and P_2, each of mass m, are joined by a light smooth inextensible string of length ℓ. P_1 lies on a table top a distance d from the edge, and P_2 hangs over the edge of the table and is suspended a distance b above the ground. The coefficient of friction between P_1 and the table top is μ, and $\mu < 1$. The system is released from rest. Show that P_1 will fall off the edge of the table if and only if

$$\mu < \frac{b}{2d - b}.$$

Suppose that $\mu > b/(2d - b)$, so that P_1 comes to rest on the table, and that the coefficient of restitution between P_2 and the floor is e. Show that, if $e > 1/(2\mu)$, then P_1 comes to rest before P_2 bounces a second time.

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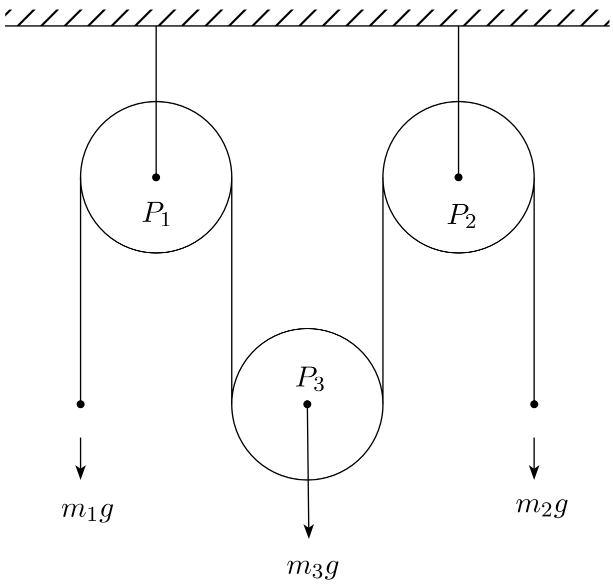

In the diagram P_1 and P_2 are smooth light pulleys fixed at the same height, and P_3 is a third smooth light pulley, freely suspended. A smooth light inextensible string runs over P_1, under P_3 and over P_2, as shown: the parts of the string not in contact with any pulley are vertical. A particle of mass m_3 is attached to P_3. There is a particle of mass m_1 attached to the end of the string below P_1 and a particle of mass m_2 attached to the other end, below P_2. The system is released from rest. Find the tension in the string, and show that the pulley P_3 will remain at rest if

$$4m_1m_2 = m_3(m_1 + m_2).$$

Section C: Probability and Statistics

15 A point moves in unit steps on the x-axis starting from the origin. At each step the point is equally likely to move in the positive or negative direction. The probability that after s steps it is at one of the points $x = 2, x = 3, x = 4$ or $x = 5$ is $\mathrm{P}(s)$. Show that $\mathrm{P}(5) = \frac{3}{16}$, $\mathrm{P}(6) = \frac{21}{64}$ and

$$\mathrm{P}(2k) = \binom{2k+1}{k-1}\left(\frac{1}{2}\right)^{2k}$$

where k is a positive integer. Find a similar expression for $\mathrm{P}(2k+1)$.

Determine the values of s for which $\mathrm{P}(s)$ has its greatest value.

16 A taxi driver keeps a packet of toffees and a packet of mints in her taxi. From time to time she takes either a toffee (with probability p) or mint (with probability $q = 1 - p$). At the beginning of the week she has n toffees and m mints in the packets. On the Nth occasion that she reaches for a sweet, she discovers (for the first time) that she has run out of that kind of sweet. What is the probability that she was reaching for a toffee?