Section A: Pure Mathematics

1 **(i)** Given that

$$f(x) = \ln(1 + e^x),$$

prove that $\ln[f'(x)] = x - f(x)$ and that $f''(x) = f'(x) - [f'(x)]^2$. Hence, or otherwise, expand $f(x)$ as a series in powers of x up to the term in x^4.

(ii) Given that

$$g(x) = \frac{1}{\sinh x \cosh 2x},$$

explain why $g(x)$ can not be expanded as a series of non-negative powers of x but that $xg(x)$ can be expanded. Explain also why this latter expansion will consist of even powers of x only. Expand $xg(x)$ as a series as far as the term in x^4.

2 The matrices $\mathbf{I}$ and $\mathbf{J}$ are

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

respectively and $\mathbf{A} = \mathbf{I} + a\mathbf{J}$, where a is a non-zero real constant. Prove that

$$\mathbf{A}^2 = \mathbf{I} + \tfrac{1}{2}[(1+2a)^2 - 1]\mathbf{J} \quad \text{and} \quad \mathbf{A}^3 = \mathbf{I} + \tfrac{1}{2}[(1+2a)^3 - 1]\mathbf{J}$$

and obtain a similar form for $\mathbf{A}^4$.

If $\mathbf{A}^k = \mathbf{I} + p_k\mathbf{J}$, suggest a suitable form for p_k and prove that it is correct by induction, or otherwise.

3 Sketch the curve C_1 whose parametric equations are $x = t^2$, $y = t^3$.

The circle C_2 passes through the origin O. The points R and S with real non-zero parameters r and s respectively are other intersections of C_1 and C_2. Show that r and s are roots of an equation of the form

$$t^4 + t^2 + at + b = 0,$$

where a and b are real constants.

By obtaining a quadratic equation, with coefficients expressed in terms of r and s, whose roots would be the parameters of any further intersections of C_1 and C_2, or otherwise, show that O, R and S are the only real intersections of C_1 and C_2.

4 A set of curves S_1 is defined by the equation

$$y = \frac{x}{x-a},$$

where a is a constant which is different for different members of S_1. Sketch on the same axes the curves for which $a = -2, -1, 1$ and 2.

A second of curves S_2 is such that at each intersection between a member of S_2 and a member of S_1 the tangents of the intersecting curves are perpendicular. On the same axes as the already sketched members of S_1, sketch the member of S_2 that passes through the point $(1, -1)$.

Obtain the first order differential equation for y satisfied at all points on all members of S_1 (i.e. an equation connecting x, y and $\mathrm{d}y/\mathrm{d}x$ which does not involve a).

State the relationship between the values of $\mathrm{d}y/\mathrm{d}x$ on two intersecting curves, one from S_1 and one from S_2, at their intersection. Hence show that the differential equation for the curves of S_2 is

$$x = y(y-1)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Find an equation for the member of S_2 that you have sketched.

5 The tetrahedron $ABCD$ has A at the point $(0, 4, -2)$. It is symmetrical about the plane $y + z = 2$, which passes through A and D. The mid-point of BC is N. The centre, Y, of the sphere $ABCD$ is at the point $(3, -2, 4)$ and lies on AN such that $\overrightarrow{AY} = 3\overrightarrow{YN}$. Show that $BN = 6\sqrt{2}$ and find the coordinates of B and C.

The angle AYD is $\cos^{-1}\frac{1}{3}$. Find the coordinates of D. [There are two alternative answers for each point.]

6 Given that $I_n = \displaystyle\int_0^{\pi} \frac{x\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x$, where n is a positive integer, show that $I_n - I_{n-1} = J_n$, where

$$J_n = \int_0^{\pi} \frac{x\sin(2n-1)x}{\sin x}\,\mathrm{d}x.$$

Obtain also a reduction formula for J_n.

The curve C is given by the cartesian equation

$$y = \frac{x\sin^2(nx)}{\sin^2 x},$$

where n is a positive integer and $0 \leqslant x \leqslant \pi$. Show that the area under the curve C is $\frac{1}{2}n\pi^2$.

7 The points P and R lie on the sides AB and AD, respectively, of the parallelogram $ABCD$. The point Q is the fourth vertex of the parallelogram $APQR$. Prove that BR, CQ and DP meet in a point.

8 Show that

$$\sin(2n+1)\theta = \sin^{2n+1}\theta \sum_{r=0}^{n} (-1)^{n-r} \binom{2n+1}{2r} \cot^{2r}\theta,$$

where n is a positive integer. Deduce that the equation

$$\sum_{r=0}^{n} (-1)^{r} \binom{2n+1}{2r} x^{r} = 0$$

has roots $\cot^2(k\pi/(2n+1))$ for $k = 1, 2, \ldots, n$.

Show that

(i) $\displaystyle\sum_{k=1}^{n} \cot^2\left(\frac{k\pi}{2n+1}\right) = \frac{n(2n-1)}{3},$

(ii) $\displaystyle\sum_{k=1}^{n} \tan^2\left(\frac{k\pi}{2n+1}\right) = n(2n+1),$

(iii) $\displaystyle\sum_{k=1}^{n} \operatorname{cosec}^2\left(\frac{k\pi}{2n+1}\right) = \frac{2n(n+1)}{3}.$

9 The straight line OSA, where O is the origin, bisects the angle between the positive x and y axes. The ellipse E has S as focus. In polar coordinates with S as pole and SA as the initial line, E has equation $\ell = r(1 + e\cos\theta)$. Show that, at the point on E given by $\theta = \alpha$, the gradient of the tangent to the ellipse is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin\alpha - \cos\alpha - e}{\sin\alpha + \cos\alpha + e}.$$

The points on E given by $\theta = \alpha$ and $\theta = \beta$ are the ends of a diameter of E. Show that

$$\tan(\alpha/2)\tan(\beta/2) = -\frac{1+e}{1-e}.$$

[**Hint.** A diameter of an ellipse is a chord through its centre.]

10 Sketch the curve C whose polar equation is

$$r = 4a\cos 2\theta \qquad \text{for } -\tfrac{1}{4}\pi < \theta < \tfrac{1}{4}\theta.$$

The ellipse E has parametric equations

$$x = 2a\cos\phi, \qquad y = a\sin\phi.$$

Show, without evaluating the integrals, that the perimeters of C and E are equal.

Show also that the areas of the regions enclosed by C and E are equal.

Section B: Mechanics

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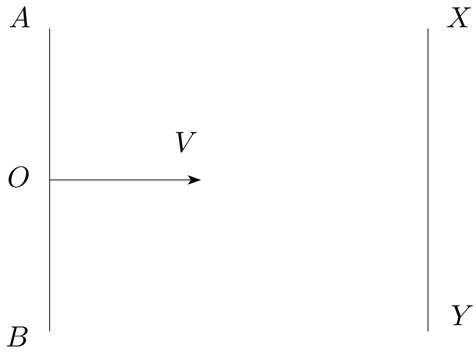

AOB represents a smooth vertical wall and XY represents a parallel smooth vertical barrier, both standing on a smooth horizontal table. A particle P is projected along the table from O with speed V in a direction perpendicular to the wall. At the time of projection, the distance between the wall and the barrier is $(75/32)VT$, where T is a constant. The barrier moves directly towards the wall, remaining parallel to the wall, with initial speed $4V$ and with constant acceleration $4V/T$ directly away from the wall. The particle strikes the barrier XY and rebounds. Show that this impact takes place at time $5T/8$.

The barrier is sufficiently massive for its motion to be unaffected by the impact. Given that the coefficient of restitution is $1/2$, find the speed of P immediately after impact.

P strikes AB and rebounds. Given that the coefficient of restitution for this collision is also $1/2,$ show that the next collision of P with the barrier is at time $9T/8$ from the start of the motion.

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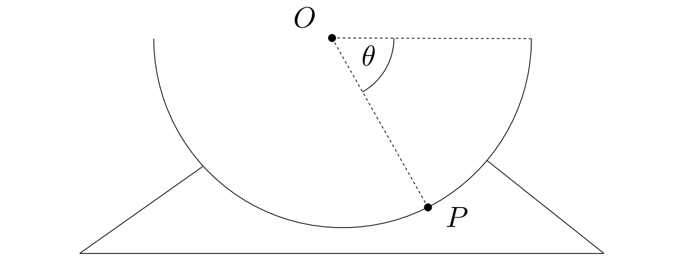

A smooth hemispherical bowl of mass $2m$ is rigidly mounted on a light carriage which slides freely on a horizontal table as shown in the diagram. The rim of the bowl is horizontal and has centre O. A particle P of mass m is free to slide on the inner surface of the bowl. Initially, P is in contact with the rim of the bowl and the system is at rest. The system is released and when OP makes an angle θ with the horizontal the velocity of the bowl is v? Show that

$$3v = a\dot{\theta}\sin\theta$$

and that

$$v^2 = \frac{2ga\sin^3\theta}{3(3-\sin^2\theta)},$$

where a is the interior radius of the bowl.

Find, in terms of m, g and $\theta,$ the reaction between the bowl and the particle.

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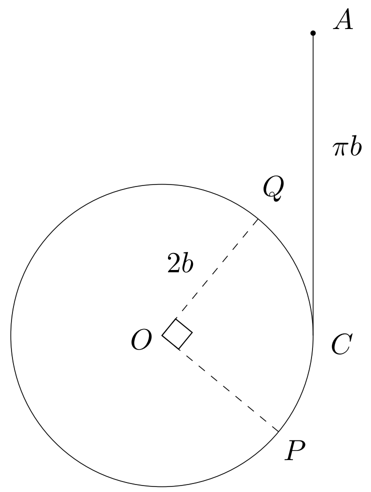

A uniform circular disc of radius $2b$, mass m and centre O is free to turn about a fixed horizontal axis through O perpendicular to the plane of the disc. A light elastic string of modulus kmg, where $k > 4/\pi$, has one end attached to a fixed point A and the other end to the rim of the disc at P. The string is in contact with the rim of the disc along the arc PC, and OC is horizontal. The natural length of the string and the length of the line AC are each πb and AC is vertical. A particle Q of mass m is attached to the rim of the disc and $\angle POQ = 90°$ as shown in the diagram. The system is released from rest with OP vertical and P below O. Show that P reaches C and that then the upward vertical component of the reaction on the axis is $mg(10 - \pi k)/3$.

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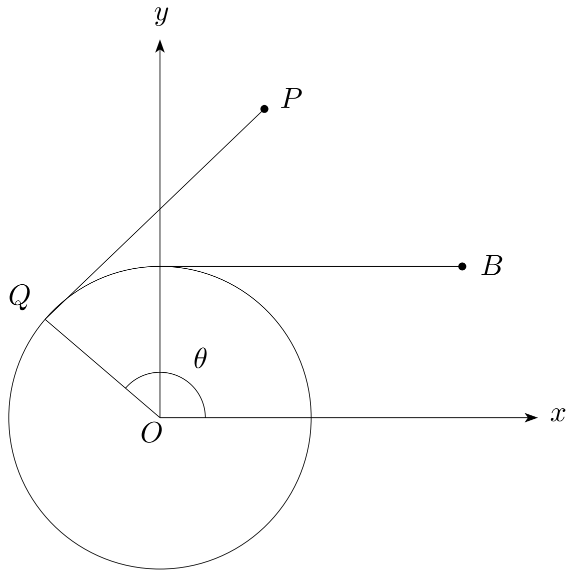

A horizontal circular disc of radius a and centre O lies on a horizontal table and is fixed to it so that it cannot rotate. A light inextensible string of negligible thickness is wrapped round the disc and attached at its free end to a particle P of mass m. When the string is all in contact with the disc, P is at A. The string is unwound so that the part not in contact with the disc is taut and parallel to OA. P is then at B. The particle is projected along the table from B with speed V perpendicular to and away from OA. In the general position, the string is tangential to the disc at Q and $\angle AOQ = \theta$. Show that, in the general position, the x-coordinate of P with respect to the axes shown in the figure is $a\cos\theta + a\theta\sin\theta$, and find y-coordinate of P. Hence, or otherwise, show that the acceleration of P has components $a\theta\dot{\theta}^2$ and $a\dot{\theta}^2 + a\theta\ddot{\theta}$ along and perpendicular to PQ, respectively.

The friction force between P and the table is $2\lambda m v^2/a$, where v is the speed of P and λ is a constant. Show that

$$\frac{\ddot{\theta}}{\dot{\theta}} = -\left(\frac{1}{\theta} + 2\lambda\theta\right)\dot{\theta}$$

and find $\dot{\theta}$ in terms of θ, λ and a. Find also the tension in the string when $\theta = \pi$.

Section C: Probability and Statistics

15 A goat G lies in a square field $OABC$ of side a. It wanders randomly round its field, so that at any time the probability of its being in any given region is proportional to the area of this region. Write down the probability that its distance, R, from O is less than r if $0 < r \leqslant a$, and show that if $r \geqslant a$ the probability is

$$\left(\frac{r^2}{a^2} - 1\right)^{\frac{1}{2}} + \frac{\pi r^2}{4a^2} - \frac{r^2}{a^2}\cos^{-1}\left(\frac{a}{r}\right).$$

Find the median of R and probability density function of R.

The goat is then tethered to the corner O by a chain of length a. Find the conditional probability that its distance from the fence OC is more than $a/2$.

16 The probability that there are exactly n misprints in an issue of a newspaper is $\mathrm{e}^{-\lambda}\lambda^n/n!$ where λ is a positive constant. The probability that I spot a particular misprint is p, independent of what happens for other misprints, and $0 < p < 1$.

(i) If there are exactly $m+n$ misprints, what is the probability that I spot exactly m of them?

(ii) Show that, if I spot exactly m misprints, the probability that I have failed to spot exactly n misprints is

$$\frac{(1-p)^n\lambda^n}{n!}\mathrm{e}^{-(1-p)\lambda}.$$