Section A: Pure Mathematics

1 I have two dice whose faces are all painted different colours. I number the faces of one of them $1, 2, 2, 3, 3, 6$ and the other $1, 3, 3, 4, 5, 6$. I can now throw a total of 3 in two different ways using the two number 2's on the first die once each. Show that there are seven different ways of throwing a total of 6.

I now renumber the dice (again only using integers in the range 1 to 6) with the results shown in the following table

Total shown by the two dice	2	3	4	5	6	7	8	9	10	11	12
Different ways of obtaining the total	0	2	1	1	4	3	8	6	5	6	0

Find how I have numbered the dice explaining your reasoning.

[You will only get high marks if the examiner can follow your argument.]

2 If $|r| \neq 1$, show that

$$1 + r^2 + r^4 + \cdots + r^{2n} = \frac{1 - r^{2n+2}}{1 - r^2}\,.$$

If $r \neq 1$, find an expression for $\mathrm{S}_n(r)$, where

$$\mathrm{S}_n(r) = r + r^2 + r^4 + r^5 + r^7 + r^8 + r^{10} + \cdots + r^{3n-1}.$$

Show that, if $|r| < 1$, then, as $n \to \infty$,

$$\mathrm{S}_n(r) \to \frac{1}{1-r} - \frac{1}{1-r^3}\,.$$

If $|r| \neq 1$, find an expression for $\mathrm{T}_n(r)$, where

$$\mathrm{T}_n(r) = 1 + r^2 + r^3 + r^4 + r^6 + r^8 + r^9 + r^{10} + r^{12} + r^{14} + r^{15} + r^{16} + \cdots + r^{6n}.$$

If $|r| < 1$, find the limit of $\mathrm{T}_n(r)$ as $n \to \infty$.

What happens to $\mathrm{T}_n(r)$ as $n \to \infty$ in the three cases $r > 1, r = 1$ and $r = -1$? In each case give reasons for your answer.

3 **(i)** Find all the integer solutions with $1 \leqslant p \leqslant q \leqslant r$ of the equation

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1\,,$$

showing that there are no others.

(ii) The integer solutions with $1 \leqslant p \leqslant q \leqslant r$ of

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1\,,$$

include $p = 1$, $q = n$, $r = m$ where n and m are any integers satisfying $1 \leqslant m \leqslant n$. Find all the other solutions, showing that you have found them all.

4 By making the change of variable $t = \pi - x$ in the integral

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x,$$

or otherwise, show that, for any function f,

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \mathrm{f}(\sin x)\,\mathrm{d}x\,.$$

Evaluate

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x \quad \text{and} \quad \int_0^{2\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x\,.$$

5 If $z = x + \mathrm{i}y$ where x and y are real, define $|z|$ in terms of x and y. Show, using your definition, that if $z_1, z_2 \in \mathbb{C}$ then $|z_1 z_2| = |z_1|\,|z_2|$.

Explain, by means of a diagram, or otherwise, why $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

Suppose that $a_j \in \mathbb{C}$ and $|a_j| \leqslant 1$ for $j = 1, 2, \ldots, n$. Show that, if $|z| \leqslant \frac{1}{2}$, then

$$\left|a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z\right| < 1,$$

and deduce that any root w of the equation

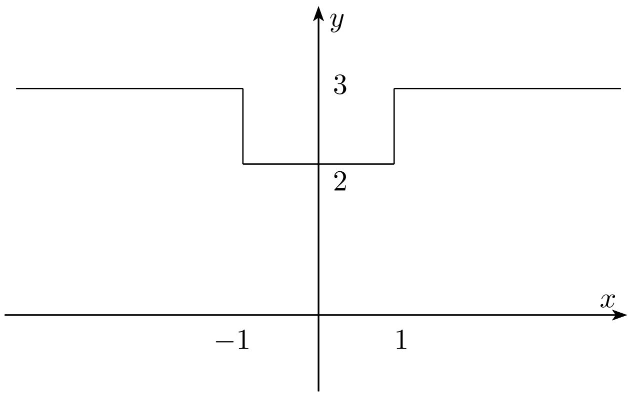
$$a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + 1 = 0$$

must satisfy $|x| > \frac{1}{2}$.

6 Let $N = 10^{100}$. The graph of

$$\mathrm{f}(x) = \frac{x^N}{1+x^N} + 2$$

for $-3 \leqslant x \leqslant 3$ is sketched in the following diagram.



Explain the main features of the sketch.

Sketch the graphs for $-3 \leqslant x \leqslant 3$ of the two functions

$$\mathrm{g}(x) = \frac{x^{N+1}}{1+x^N}$$

and

$$\mathrm{h}(x) = 10^N \sin(10^{-N} x).$$

In each case explain briefly the main features of your sketch.

7 Sketch the curve

$$\mathrm{f}(x) = x^3 + Ax^2 + B$$

first in the case $A > 0$ and $B > 0$, and then in the case $A < 0$ and $B > 0$.

Show that the equation

$$x^3 + ax^2 + b = 0,$$

where a and b are real, will have three distinct real roots if

$$27b^2 + 3a^3b < 0,$$

but will have fewer than three if

$$27b^2 + 4a^3b < 0.$$

8 **(i)** Prove that the intersection of the surface of a sphere with a plane is always a circle, a point or the empty set. Prove that the intersection of the surfaces of two spheres with distinct centres is always a circle, a point or the empty set.

[If you use coordinate geometry, a careful choice of origin and axes may help.]

(ii) The parish council of Little Fitton have just bought a modern sculpture entitled 'Truth, Love and Justice pouring forth their blessings on Little Fitton.' It consists of three vertical poles AD, BE and CF of heights 2 metres, 3 metres and 4 metres respectively. Show that $\angle DEF = \cos^{-1}\frac{1}{5}$.

Vandals now shift the pole AD so that A is unchanged and the pole is still straight but D is vertically above AB with $\angle BAD = \frac{1}{4}\pi$ (in radians). Find the new angle $\angle DEF$ in radians correct to four figures.

9 In the manufacture of Grandma's Home Made Ice-cream, chemicals A and B pour at constant rates a and $b - a$ litres per second $(0 < a < b)$ into a mixing vat which mixes the chamicals rapidly and empties at a rate b litres per second into a second mixing vat. At time $t = 0$ the first vat contains K litres of chemical B only. Show that the volume $V(t)$ (in litres) of the chemical A in the first vat is governed by the differential equation

$$\dot{V}(t) = -\frac{bV(t)}{K} + a,$$

and that

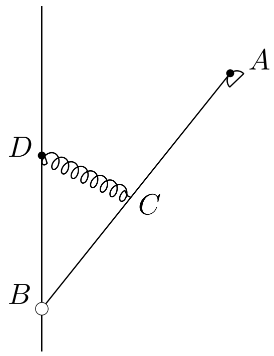
$$V(t) = \frac{aK}{b}(1 - \mathrm{e}^{-bt/K})$$

for $t \geqslant 0$.

The second vat also mixes chemicals rapidly and empties at the rate of b litres per second. If at time $t = 0$ it contains L litres of chemical C only (where $L \neq K$), how many litres of chemical A will it contain at a later time t?

Section B: Mechanics

10 A small lamp of mass m is at the end A of a light rod AB of length $2a$ attached at B to a vertical wall in such a way that the rod can rotate freely about B in a vertical plane perpendicular to the wall. A spring CD of natural length a and modulus of elasticity λ is joined to the rod at its mid-point C and to the wall at a point D a distance a vertically above B. The arrangement is sketched below.



Show that if $\lambda > 4mg$ the lamp can hang in equilibrium away from the wall and calculate the angle $\angle DBA$.

11 A piece of uniform wire is bent into three sides of a square $ABCD$ so that the side AD is missing. Show that if it is first hung up by the point A and then by the point B then the angle between the two directions of BC is $\tan^{-1} 18$.

12 In a clay pigeon shoot the target is launched vertically from ground level with speed v. At a time T later the competitor fires a rifle inclined at angle α to the horizontal. The competitor is also at ground level and is a distance l from the launcher. The speed of the bullet leaving the rifle is u. Show that, if the competitor scores a hit, then

$$l \sin\alpha - \left(vT - \tfrac{1}{2}gT^2\right)\cos\alpha = \frac{v - gT}{u} l.$$

Suppose now that $T = 0$. Show that if the competitor can hit the target before it hits the ground then $v < u$ and

$$\frac{2v\sqrt{u^2 - v^2}}{g} > l.$$

13 A train starts from a station. The tractive force exerted by the engine is at first constant and equal to F. However, after the speed attains the value u, the engine works at constant rate P, where $P = Fu$. The mass of the engine and the train together is M. Forces opposing motion may be neglected. Show that the engine will attain a speed v, with $v \geqslant u$, after a time

$$t = \frac{M}{2P}\left(u^2 + v^2\right).$$

Show also that it will have travelled a distance

$$\frac{M}{6P}(2v^3 + u^2)$$

in this time.

Section C: Probability and Statistics

14 When he sets out on a drive Mr Toad selects a speed V kilometres per minute where V is a random variable with probability density

$$\alpha v^{-2} \mathrm{e}^{-\alpha v^{-1}}$$

and α is a strictly positive constant. He then drives at constant speed, regardless of other drivers, road conditions and the Highway Code. The traffic lights at the Wild Wood cross-roads change from red to green when Mr Toad is exactly 1 kilometre away in his journey towards them. If the traffic light is green for g minutes, then red for r minutes, then green for g minutes, and so on, show that the probability that he passes them after $n(g+r)$ minutes but before $n(g+r)+g$ minutes, where n is a positive integer, is

$$\mathrm{e}^{-\alpha n(g+r)} - \mathrm{e}^{-\alpha(n(g+r))+g}.$$

Find the probability $\mathrm{P}(\alpha)$ that he passes the traffic lights when they are green.

Show that $\mathrm{P}(\alpha) \to 1$ as $\alpha \to \infty$ and, by noting that $(\mathrm{e}^x - 1)/x \to 1$ as $x \to 0$, or otherwise, show that

$$\mathrm{P}(\alpha) \to \frac{g}{r+g} \quad \text{as } \alpha \to 0.$$

[NB: the traffic light show only green and red - not amber.]

15 Captain Spalding is on a visit to the idyllic island of Gambriced. The population of the island consists of the two lost tribes of Frodox and the latest census shows that $11/16$ of the population belong to the Ascii who tell the truth $3/4$ of the time and $5/16$ to the Biscii who always lie. The answers of an Ascii to each question (even if it is the same as one before) are independent.

Show that the probability that an Ascii gives the same answer twice in succession to the same question is $5/8$. Show that the probability that an Ascii gives the same answer twice is telling the truth is $9/10$.

Captain Spalding addresses one of the natives as follows.

Spalding: My good man, I'm afraid I'm lost. Should I go left or right to reach the nearest town?

Native: Left.

Spalding: I am a little deaf. Should I go left or right to reach the nearest town?

Native (patiently): Left.

Show that, on the basis of this conversation, Captain Spalding should go left to try and reach the nearest town and that there is a probability $99/190$ that this is the correct direction.

The conversation resumes as follows.

Spalding: I'm sorry I didn't quite hear that. Should I go left or right to reach the nearest town?

Native (loudly and clearly): Left.

Shouls Captain Spalding go left or right and why? Show that if he follows your advice the probability that this is the correct direction is $331/628$.

16 By making the substitution $y = \cos^{-1} t$, or otherwise, show that

$$\int_0^1 \cos^{-1} t \, \mathrm{d}t = 1.$$

A pin of length $2a$ is thrown onto a floor ruled with parallel lines equally spaced at a distance $2b$ apart. The distance X of its centre from the nearest line is a uniformly distributed random variable taking values between 0 and b and the acute angle Y the pin makes with a direction perpendicular to the line is a uniformly distributed random variable taking values between 0 and $\pi/2$. X and Y are independent. If $X = x$ what is the probability that the pin crosses the line?

If $a < b$ show that the probability that the pin crosses a line for a general throw is $\dfrac{2a}{\pi b}$.