

**Section A: Pure Mathematics**

1 Calculate

$$\int_0^x \operatorname{sech} t \, dt.$$

Find the reduction formula involving  $I_n$  and  $I_{n-2}$ , where

$$I_n = \int_0^x \operatorname{sech}^n t \, dt$$

and, hence or otherwise, find  $I_5$  and  $I_6$ .

2 (i) By setting  $y = x + x^{-1}$ , find the solutions of

$$x^4 + 10x^3 + 26x^2 + 10x + 1 = 0.$$

(ii) Solve

$$x^4 + x^3 - 10x^2 - 4x + 16 = 0.$$

3 Describe geometrically the possible intersections of a plane with a sphere.

Let  $P_1$  and  $P_2$  be the planes with equations

$$3x - y - 1 = 0,$$

$$x - y + 1 = 0,$$

respectively, and let  $S_1$  and  $S_2$  be the spheres with equations

$$x^2 + y^2 + z^2 = 7,$$

$$x^2 + y^2 + z^2 - 6y - 4z + 10 = 0,$$

respectively. Let  $C_1$  be the intersection of  $P_1$  and  $S_1$ , let  $C_2$  be the intersection of  $P_2$  and  $S_2$  and let  $L$  be the intersection of  $P_1$  and  $P_2$ . Find the points where  $L$  meets each of  $S_1$  and  $S_2$ . Determine, giving your reasons, whether the circles  $C_1$  and  $C_2$  are linked.

- 4 Find the two solutions of the differential equation

$$\left(\frac{dy}{dx}\right)^2 = 4y$$

which pass through the point  $(a, b^2)$ , where  $b \neq 0$ .

Find two distinct points  $(a_1, 1)$  and  $(a_2, 1)$  such that one of the solutions through each of them also passes through the origin. Show that the graphs of these two solutions coincide and sketch their common graph, together with the other solutions through  $(a_1, 1)$  and  $(a_2, 1)$ .

Now sketch sufficient members of the family of solutions (for varying  $a$  and  $b$ ) to indicate the general behaviour. Use your sketch to identify a common tangent, and comment briefly on its relevance to the differential equation.

- 5 The function  $f$  is given by  $f(x) = \sin^{-1} x$  for  $-1 < x < 1$ . Prove that

$$(1 - x^2)f''(x) - xf'(x) = 0.$$

Prove also that

$$(1 - x^2)f^{(n+2)}(x) - (2n + 1)xf^{(n+1)}(x) - n^2f^{(n)}(x) = 0,$$

for all  $n > 0$ , where  $f^{(n)}$  denotes the  $n$ th derivative of  $f$ . Hence express  $f(x)$  as a Maclaurin series.

The function  $g$  is given by

$$g(x) = \ln \sqrt{\frac{1+x}{1-x}},$$

for  $-1 < x < 1$ . Write down a power series expression for  $g(x)$ , and show that the coefficient of  $x^{2n+1}$  is greater than that in the expansion of  $f$ , for each  $n > 0$ .

- 6 The four points  $A, B, C, D$  in the Argand diagram (complex plane) correspond to the complex numbers  $a, b, c, d$  respectively. The point  $P_1$  is mapped to  $P_2$  by rotating about  $A$  through  $\pi/2$  radians. Then  $P_2$  is mapped to  $P_3$  by rotating about  $B$  through  $\pi/2$  radians,  $P_3$  is mapped to  $P_4$  by rotating about  $C$  through  $\pi/2$  radians and  $P_4$  is mapped to  $P_5$  by rotating about  $D$  through  $\pi/2$  radians, each rotation being in the positive sense. If  $z_i$  is the complex number corresponding to  $P_i$ , find  $z_5$  in terms of  $a, b, c, d$  and  $z_1$ .

Show that  $P_5$  will coincide with  $P_1$ , irrespective of the choice of the latter if, and only if

$$a - c = i(b - d)$$

and interpret this condition geometrically.

The points  $A, B$  and  $C$  are now chosen to be distinct points on the unit circle and the angle of rotation is changed to  $\theta$ , where  $\theta \neq 0$ , on each occasion. Find the necessary and sufficient condition on  $\theta$  and the points  $A, B$  and  $C$  for  $P_4$  always to coincide with  $P_1$ .

- 7 Let  $S_3$  be the group of permutations of three objects and  $Z_6$  be the group of integers under addition modulo 6. List all the elements of each group, stating the order of each element. State, with reasons, whether  $S_3$  is isomorphic with  $Z_6$ .

Let  $C_6$  be the group of 6th roots of unity. That is,  $C_6 = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}$  where  $\alpha = e^{i\pi/3}$  and the group operation is complex multiplication. Prove that  $C_6$  is isomorphic with  $Z_6$ . Is there any (multiplicative or additive) subgroup of the complex numbers which is isomorphic with  $S_3$ ? Give a reason for your answer.

- 8 Let  $a, b, c, d, p, q, r$  and  $s$  be real numbers. By considering the determinant of the matrix product

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} \begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix},$$

where  $z_1, z_2, z_3$  and  $z_4$  are suitably chosen complex numbers, find expressions  $L_1, L_2, L_3$  and  $L_4$ , each of which is linear in  $a, b, c$  and  $d$  and also linear in  $p, q, r$  and  $s$ , such that

$$(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = L_1^2 + L_2^2 + L_3^2 + L_4^2.$$

## Section B: Mechanics

- 9 A smooth, axially symmetric bowl has its vertical cross-sections determined by  $s = 2\sqrt{ky}$ , where  $s$  is the arc-length measured from its lowest point  $V$ , and  $y$  is the height above  $V$ . A particle is released from rest at a point on the surface at a height  $h$  above  $V$ . Explain why

$$\left(\frac{ds}{dt}\right)^2 + 2gy$$

is constant.

Show that the time for the particle to reach  $V$  is

$$\pi\sqrt{\frac{k}{2g}}.$$

Two elastic particles of mass  $m$  and  $\alpha m$ , where  $\alpha < 1$ , are released simultaneously from opposite sides of the bowl at heights  $\alpha^2 h$  and  $h$  respectively. If the coefficient of restitution between the particles is  $\alpha$ , describe the subsequent motion.

- 10 The island of Gammaland is totally flat and subject to a constant wind of  $w \text{ kh}^{-1}$ , blowing from the West. Its southernmost shore stretches almost indefinitely, due east and west, from the coastal city of Alphabet. A novice pilot is making her first solo flight from Alphaport to the town of Betaville which lies north-east of Alphaport. Her instructor has given her the correct heading to reach Betaville, flying at the plane's recommended airspeed of  $v \text{ kh}^{-1}$ , where  $v > w$ .

On reaching Betaport the pilot returns with the opposite heading to that of the outward flight and, so featureless is Gammaland, that she only realises her error as she crosses the coast with Alphaport nowhere in sight. Assuming that she then turns West along the coast, and that her outward flight took  $t$  hours, show that her return flight takes

$$\left(\frac{v+w}{v-w}\right)t \text{ hours.}$$

If Betaville is  $d$  kilometres from Alphaport, show that, with the correct heading, the return flight should have taken

$$t + \frac{\sqrt{2}wd}{v^2 - w^2} \text{ hours.}$$

- 11** A step-ladder has two sections  $AB$  and  $AC$ , each of length  $4a$ , smoothly hinged at  $A$  and connected by a light elastic rope  $DE$ , of natural length  $a/4$  and modulus  $W$ , where  $D$  is on  $AB$ ,  $E$  is on  $AC$  and  $AD = AE = a$ . The section  $AB$ , which contains the steps, is uniform and of weight  $W$  and the weight of  $AC$  is negligible.

The step-ladder rests on a smooth horizontal floor and a man of weight  $4W$  carefully ascends it to stand on a rung distant  $\beta a$  from the end of the ladder resting on the floor. Find the height above the floor of the rung on which the man is standing when  $\beta$  is the maximum value at which equilibrium is possible.

## Section C: Probability and Statistics

- 12** In certain forms of Tennis two players  $A$  and  $B$  serve alternate games. Player  $A$  has probability  $p_A$  of winning a game in which she serves and  $p_B$  of winning a game in which player  $B$  serves. Player  $B$  has probability  $q_B = 1 - p_B$  of winning a game in which she serves and probability  $q_A = 1 - p_A$  of winning a game in which player  $A$  serves. In Shortened Tennis the first player to lead by 2 games wins the match. Find the probability  $P_{\text{short}}$  that  $A$  wins a Shortened Tennis match in which she serves first and show that it is the same as if  $B$  serves first.

In Standard Tennis the first player to lead by 2 or more games after 4 or more games have been played wins the match. Show that the probability that the match is decided in 4 games is

$$p_A^2 p_B^2 + q_A^2 q_B^2 + 2(p_A p_B + q_A q_B)(p_A q_B + q_A p_B).$$

If  $p_A = p_B = p$  and  $q_A = q_B = q$ , find the probability  $P_{\text{stan}}$  that  $A$  wins a Standard Tennis match in which she serves first. Show that

$$P_{\text{stan}} - P_{\text{short}} = \frac{p^2 q^2 (p - q)}{p^2 + q^2}.$$

- 13** During his performance a trapeze artist is supported by two identical ropes, either of which can bear his weight. Each rope is such that the time, in hours of performance, before it fails is exponentially distributed, independently of the other, with probability density function  $\lambda \exp(-\lambda t)$  for  $t \geq 0$  (and 0 for  $t < 0$ ), for some  $\lambda > 0$ . A particular rope has already been in use for  $t_0$  hours of performance. Find the distribution for the length of time the artist can continue to use it before it fails. Interpret and comment upon your result.

Before going on tour the artist insists that the management purchase two new ropes of the above type. Show that the probability density function of the time until both ropes fail is

$$f(t) = \begin{cases} 2\lambda e^{-\lambda t}(1 - e^{-\lambda t}) & \text{if } t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If each performance lasts for  $h$  hours, find the probability that both ropes fail during the  $n$ th performance. Show that the probability that both ropes fail during the same performance is  $\tanh(\lambda h/2)$ .

- 14** Three points,  $P, Q$  and  $R$ , are independently randomly chosen on the perimeter of a circle. Prove that the probability that at least one of the angles of the triangle  $PQR$  will exceed  $k\pi$  is  $3(1 - k)^2$  if  $\frac{1}{2} \leq k \leq 1$ . Find the probability if  $\frac{1}{3} \leq k \leq \frac{1}{2}$ .