Section A: Pure Mathematics

1 A cylindrical biscuit tin has volume V and surface area S (including the ends). Show that the minimum possible surface area for a given value of V is $S = 3(2\pi V^2)^{1/3}$. For this value of S show that the volume of the largest sphere which can fit inside the tin is $\frac{2}{3}V$, and find the volume of the smallest sphere into which the tin fits.

2 **(i)** Show that

$$\int_0^1 (1 + (\alpha - 1)x)^n \, \mathrm{d}x = \frac{\alpha^{n+1} - 1}{(n+1)(\alpha - 1)}$$

when $\alpha \neq 1$ and n is a positive integer.

(ii) Show that if $0 \leqslant k \leqslant n$ then the coefficient of α^k in the polynomial

$$\int_0^1 (\alpha x + (1 - x))^n \, \mathrm{d}x$$

is

$$\binom{n}{k} \int_0^1 x^k (1 - x)^{n-k} \, \mathrm{d}x \, .$$

(iii) Hence, or otherwise, show that

$$\int_0^1 x^k (1 - x)^{n-k} \, \mathrm{d}x = \frac{k!(n-k)!}{(n+1)!} \, .$$

3 Let n be a positive integer.

(i) Factorise $n^5 - n^3$, and show that it is divisible by 24.

(ii) Prove that $2^{2n} - 1$ is divisible by 3.

(iii) If $n - 1$ is divisible by 3, show that $n^3 - 1$ is divisible by 9.

4 Show that

$$\int_0^1 \frac{1}{x^2+2ax+1}\,\mathrm{d}x = \begin{cases} \dfrac{1}{\sqrt{1-a^2}}\tan^{-1}\sqrt{\dfrac{1-a}{1+a}} & \text{if } |a|<1, \\ \dfrac{1}{2\sqrt{a^2-1}}\ln\left|a+\sqrt{a^2-1}\right| & \text{if } |a|>1. \end{cases}$$

5 **(i)** Find all rational numbers r and s which satisfy

$$(r+s\sqrt{3})^2 = 4-2\sqrt{3}.$$

(ii) Find all real numbers p and q which satisfy

$$(p+q\mathrm{i})^2 = (3-2\sqrt{3}) + 2(1-\sqrt{3})\mathrm{i}.$$

(iii) Solve the equation

$$(1+\mathrm{i})z^2 - 2z + 2\sqrt{3} - 2 = 0,$$

writing your solutions in as simple a form as possible.

[No credit will be given to answers involving use of calculators.]

6 Let $\mathrm{f}(x) = \dfrac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}$ for $0 < x \leqslant \pi$.

(i) Using the formula

$$2\sin\tfrac{1}{2}x\cos kx = \sin(k+\tfrac{1}{2})x - \sin(k-\tfrac{1}{2})x$$

(which you may assume), or otherwise, show that

$$\mathrm{f}(x) = 1 + 2\sum_{k=1}^{n}\cos kx\,.$$

(ii) Find $\displaystyle\int_0^\pi \mathrm{f}(x)\,\mathrm{d}x$ and $\displaystyle\int_0^\pi \mathrm{f}(x)\cos x\,\mathrm{d}x$.

7 **(i)** At time $t = 0$ a tank contains one unit of water. Water flows out of the tank at a rate proportional to the amount of water in the tank. The amount of water in the tank at time t is y. Show that there is a constant $b < 1$ such that $y = b^t$.

(ii) Suppose instead that the tank contains one unit of water at time $t = 0$, but that in addition to water flowing out as described, water is added at a steady rate $a > 0$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}t} - y \ln b = a,$$

and hence find y in terms of a, b and t.

8 **(i)** By using the formula for the sum of a geometric series, or otherwise, express the number $0.38383838\ldots$ as a fraction in its lowest terms.

(ii) Let x be a real number which has a recurring decimal expansion

$$x = 0 \cdot a_1 a_2 a_2 \cdots ,$$

so that there exists positive integers N and k such that $a_{n+k} = a_n$ for all $n > N$. Show that

$$x = \frac{b}{10^N} + \frac{c}{10^N(10^k - 1)} ,$$

where b and c are integers to be found. Deduce that x is rational.

Section B: Mechanics

9 A bungee-jumper of mass m is attached by means of a light rope of natural length l and modulus of elasticity mg/k, where k is a constant, to a bridge over a ravine. She jumps from the bridge and falls vertically towards the ground. If she only just avoids hitting the ground, show that the height h of the bridge above the floor of the ravine satisfies

$$h^2 - 2hl(k+1) + l^2 = 0,$$

and hence find h. Show that the maximum speed v which she attains during her fall satisfies

$$v^2 = (k+2)gl.$$

10 A spaceship of mass M is at rest. It separates into two parts in an explosion in which the total kinetic energy released is E. Immediately after the explosion the two parts have masses m_1 and m_2 and speeds v_1 and v_2 respectively. Show that the minimum possible relative speed $v_1 + v_2$ of the two parts of the spaceship after the explosion is $(8E/M)^{1/2}$.

11 A particle is projected under the influence of gravity from a point O on a level plane in such a way that, when its horizontal distance from O is c, its height is h. It then lands on the plane at a distance $c+d$ from O. Show that the angle of projection α satisfies

$$\tan\alpha = \frac{h(c+d)}{cd}$$

and that the speed of projection v satisfies

$$v^2 = \frac{g}{2}\left(\frac{cd}{h} + \frac{(c+d)^2 h}{cd}\right).$$

Section C: Probability and Statistics

12 An examiner has to assign a mark between 1 and m inclusive to each of n examination scripts ($n \leqslant m$). He does this randomly, but never assigns the same mark twice. If K is the highest mark that he assigns, explain why

$$\mathrm{P}(K = k) = \binom{k-1}{n-1} \bigg/ \binom{m}{n}$$

for $n \leqslant k \leqslant m$, and deduce that

$$\sum_{k=n}^{m} \binom{k-1}{n-1} = \binom{m}{n}.$$

Find the expected value of K.

13 I have a Penny Black stamp which I want to sell to my friend Jim, but we cannot agree a price. So I put the stamp under one of two cups, jumble them up, and let Jim guess which one it is under. If he guesses correctly, I add a third cup, jumble them up, and let Jim guess correctly, adding another cup each time. The price he pays for the stamp is $£N$, where N is the number of cups present when Jim fails to guess correctly. Find $\mathrm{P}(N = k)$. Show that $\mathrm{E}(N) = \mathrm{e}$ and calculate $\mathrm{Var}(N)$.

14 A biased coin, with a probability p of coming up heads and a probability $q = 1 - p$ of coming up tails, is tossed repeatedly. Let A be the event that the first run of r successive heads occurs before the first run of s successive tails. If H is the even that on the first toss the coin comes up heads and T is the event that it comes up tails, show that

$$\mathrm{P}(A|H) = p^{\alpha} + (1 - p^{\alpha})\mathrm{P}(A|T),$$
$$\mathrm{P}(A|T) = (1 - q^{\beta})\mathrm{P}(A|H),$$

where α and β are to be determined. Use these two equations to find $\mathrm{P}(A|H)$, $\mathrm{P}(A|T)$, and hence $\mathrm{P}(A)$.