Section A: Pure Mathematics

1 Define $\cosh x$ and $\sinh x$ in terms of exponentials and prove, from your definitions, that

$$\cosh^4 x - \sinh^4 x = \cosh 2x$$

and

$$\cosh^4 x + \sinh^4 x = \tfrac{1}{4}\cosh 4x + \tfrac{3}{4}.$$

Find $a_0, a_1, \ldots, a_n$ in terms of n such that

$$\cosh^n x = a_0 + a_1 \cosh x + a_2 \cosh 2x + \cdots + a_n \cosh nx.$$

Hence, or otherwise, find expressions for $\cosh^{2m} x - \sinh^{2m} x$ and $\cosh^{2m} x + \sinh^{2m} x$, in terms of $\cosh kx$, where $k = 0, \ldots, 2m$.

2 For all values of a and b, either solve the simultaneous equations

$$\begin{aligned} x + y + az &= 2 \\ x + ay + z &= 2 \\ 2x + y + z &= 2b \end{aligned}$$

or prove that they have no solution.

3 Find

$$\int_0^{\theta} \frac{1}{1 - a\cos x}\,\mathrm{d}x\,,$$

where $0 < \theta < \pi$ and $-1 < a < 1$.

Hence show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{2 - a\cos x}\,\mathrm{d}x = \frac{2}{\sqrt{4 - a^2}}\tan^{-1}\sqrt{\frac{2 + a}{2 - a}}\,,$$

and also that

$$\int_0^{\frac{3}{4}\pi} \frac{1}{\sqrt{2} + \cos x}\,\mathrm{d}x = \frac{\pi}{2}\,.$$

4 Find the integers k satisfying the inequality $k \leqslant 2(k-2)$.

Given that N is a strictly positive integer consider the problem of finding strictly positive integers whose sum is N and whose product is as large as possible. Call this largest possible product $P(N)$. Show that $P(5) = 2\times 3$, $P(6) = 3^2$, $P(7) = 2^2\times 3$, $P(8) = 2\times 3^2$ and $P(9) = 3^3$.

Find $P(1000)$ explaining your reasoning carefully.

5 Show, using de Moivre's theorem, or otherwise, that

$$\tan 7\theta = \frac{t(t^6 - 21t^4 + 35t^2 - 7)}{7t^6 - 35t^4 + 21t^2 - 1},$$

where $t = \tan\theta$.

(i) By considering the equation $\tan 7\theta = 0$, or otherwise, obtain a cubic equation with integer coefficients whose roots are

$$\tan^2\left(\frac{\pi}{7}\right),\ \tan^2\left(\frac{2\pi}{7}\right) \text{ and } \tan^2\left(\frac{3\pi}{7}\right)$$

and deduce the value of

$$\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right).$$

(ii) Find, without using a calculator, the value of

$$\tan^2\left(\frac{\pi}{14}\right) + \tan^2\left(\frac{3\pi}{14}\right) + \tan^2\left(\frac{5\pi}{14}\right).$$

6 **(i)** Let S be the set of matrices of the form

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix},$$

where a is any real non-zero number. Show that S is closed under matrix multiplication and, further, that S is a group under matrix multiplication.

(ii) Let G be a set of $n\times n$ matrices which is a group under matrix multiplication, with identity element $\mathbf{E}$. By considering equations of the form $\mathbf{BC} = \mathbf{D}$ for suitable elements $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of G, show that if a given element $\mathbf{A}$ of G is a singular matrix (i.e. $\det \mathbf{A} = 0$), then all elements of G are singular. Give, with justification, an example of such a group of singular matrices in the case $n = 3$.

7 **(i)** If $x+y+z=\alpha$, $xy+yz+zx=\beta$ and $xyz=\gamma$, find numbers A, B and C such that

$$x^3+y^3+z^3=A\alpha^3+B\alpha\beta+C.$$

Solve the equations

$$\begin{aligned} x+y+z &= 1 \\ x^2+y^2+z^2 &= 3 \\ x^3+y^3+z^3 &= 4. \end{aligned}$$

(ii) The area of a triangle whose sides are a, b and c is given by the formula

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter $\frac{1}{2}(a+b+c)$. If a, b and c are the roots of the equation

$$x^3-16x^2+81x-128=0,$$

find the area of the triangle.

8 A transformation T of the real numbers is defined by

$$y=T(x)=\frac{ax-b}{cx-d},$$

where a, b, c, d are real numbers such that $ad \neq bc$. Find all numbers x such that $T(x)=x$. Show that the inverse operation, $x=T^{-1}(y)$ expressing x in terms of y is of the same form as T and find corresponding numbers a', b', c', d'.

Let S_r denote the set of all real numbers excluding r. Show that, if $c \neq 0$, there is a value of r such that T is defined for all $x \in S_r$ and find the image $T(S_r)$. What is the corresponding result if $c=0$?

If T_1, given by numbers a_1, b_1, c_1, d_1, and T_2, given by numbers a_2, b_2, c_2, d_2 are two such transformations, show that their composition T_3, defined by $T_3(x)=T_2(T_1(x))$, is of the same form.

Find necessary and sufficient conditions on the numbers a, b, c, d for T^2, the composition of T with itself, to be the identity. Hence, or otherwise, find transformations T_1, T_2 and their composition T_3 such that T_1^2 and T_2^2 are each the identity but T_3^2 is not.

Section B: Mechanics

9 A particle of mass m is at rest on top of a smooth fixed sphere of radius a. Show that, if the particle is given a small displacement, it reaches the horizontal plane through the centre of the sphere at a distance

$$a(5\sqrt{5} + 4\sqrt{23})/27$$

from the centre of the sphere.

[Air resistance should be neglected.]

10 Two rough solid circular cylinders, of equal radius and length and of uniform density, lie side by side on a rough plane inclined at an angle α to the horizontal, where $0 < \alpha < \pi/2$. Their axes are horizontal and they touch along their entire length. The weight of the upper cylinder is W_1 and the coefficient of friction between it and the plane is μ_1. The corresponding qunatities for the lower cylinder are W_2 and μ_2 respectively and the coefficient of friction between the two cylinders is μ. Show that for equilibrium to be possible:

(i) $W_1 \geqslant W_2$;

(ii) $\mu \geqslant \dfrac{W_1 + W_2}{W_1 - W_2}$;

(iii) $\mu_1 \geqslant \left(\dfrac{2W_1 \cot\alpha}{W_1 + W_2} - 1\right)^{-1}$.

Find the similar inequality to **(iii)** for μ_2.

11 A smooth circular wire of radius a is held fixed in a vertical plane with light elastic strings of natural length a and modulus λ attached to the upper and lower extremities, A and C respectively, of the vertical diameter. The other ends of the two strings are attached to a small ring B which is free to slide on the wire. Show that, while both strings remain taut, the equation for the motion of the ring is

$$2ma\ddot{\theta} = \lambda(\cos\theta - \sin\theta) - mg\sin\theta,$$

where θ is the angle $\angle CAB$.

Initially the system is at rest in equilibrium with $\sin\theta = \frac{3}{5}$. Deduce that $5\lambda = 24mg$.

The ring is now displaced slightly. Show that, in the ensuing motion, it will oscillate with period approximately

$$10\pi\sqrt{\frac{a}{91g}}\,.$$

Section C: Probability and Statistics

12 It has been observed that Professor Ecks proves three types of theorems: 1, those that are correct and new; 2, those that are correct, but already known; 3, those that are false. It has also been observed that, if a certain of her theorems is of type i, then her next theorem is of type j with probability p_{ij}, where p_{ij} is the entry in the ith row and jth column of the following array:

$$\begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}.$$

Let $a_i,\ i = 1, 2, 3$, be the probability that a given theorem is of type i, and let b_j be the consequent probability that the next theorem is of type j.

(i) Explain why $b_j = a_1 p_{1j} + a_2 p_{2j} + a_3 p_{3j}$.

(ii) Find values of a_1, a_2 and a_3 such that $b_i = a_i$ for $i = 1, 2, 3$.

(iii) For these values of the a_i find the probabilities q_{ij} that, if a particular theorem is of type j, then the *preceding* theorem was of type i.

13 Let X be a random variable which takes only the finite number of different possible real values $x_1, x_2, \ldots, x_n$. Define the expectation $\mathrm{E}(X)$ and the variance $\mathrm{var}(X)$ of X. Show that , if a and b are real numbers, then $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ and express $\mathrm{var}(aX + b)$ similarly in terms of $\mathrm{var}(X)$.

Let λ be a positive real number. By considering the contribution to $\mathrm{var}(X)$ of those x_i for which $|x_i - \mathrm{E}(X)| \geqslant \lambda$, or otherwise, show that

$$\mathrm{P}\left(|X - \mathrm{E}(X)| \geqslant \lambda\right) \leqslant \frac{\mathrm{var}(X)}{\lambda^2}.$$

Let k be a real number satisfying $k \geqslant \lambda$. If $|x_i - \mathrm{E}(X)| \leqslant k$ for all i, show that

$$\mathrm{P}\left(|X - \mathrm{E}(X)| \geqslant \lambda\right) \geqslant \frac{\mathrm{var}(X) - \lambda^2}{k^2 - \lambda^2}.$$

14 Whenever I go cycling I start with my bike in good working order. However if all is well at time t, the probability that I get a puncture in the small interval $(t, t + \delta t)$ is $\alpha\,\delta t$. How many punctures can I expect to get on a journey during which my total cycling time is T?

When I get a puncture I stop immediately to repair it and the probability that, if I am repairing it at time t, the repair will be completed in time $(t, t + \delta t)$ is $\beta\,\delta t$. If $p(t)$ is the probability that I am repairing a puncture at time t, write down an equation relating $p(t)$ to $p(t + \delta t)$, and derive from this a differential equation relating $p'(t)$ and $p(t)$. Show that

$$p(t) = \frac{\alpha}{\alpha + \beta}(1 - \mathrm{e}^{-(\alpha+\beta)t})$$

satisfies this differential equation with the appropriate initial condition.

Find an expression, involving α, β and T, for the time expected to be spent mending punctures during a journey of total time T. Hence, or otherwise, show that, the fraction of the journey expected to be spent mending punctures is given approximately by

$$\frac{\alpha T}{2} \quad \text{if } (\alpha + \beta)T \text{ is small,}$$

and by

$$\frac{\alpha}{\alpha + \beta} \quad \text{if } (\alpha + \beta)T \text{ is large.}$$