Section A: Pure Mathematics

1 Show that you can make up 10 pence in eleven ways using 10p, 5p, 2p and 1p coins.

In how many ways can you make up 20 pence using 20p, 10p, 5p, 2p and 1p coins?

[You are reminded that no credit will be given for unexplained answers.]

2 **(i)** If

$$\mathrm{f}(x) = \tan^{-1} x + \tan^{-1}\left(\frac{1-x}{1+x}\right),$$

find $\mathrm{f}'(x)$. Hence, or otherwise, find a simple expression for $\mathrm{f}(x)$.

(ii) Suppose that y is a function of x with $0 < y < (\pi/2)^{1/2}$ and

$$x = y \sin y^2$$

for $0 < x < (\pi/2)^{1/2}$. Show that (for this range of x)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x + 2y^2\sqrt{y^2 - x^2}}.$$

3 Let $a_1 = 3$, $a_{n+1} = a_n^3$ for $n \geqslant 1$. (Thus $a_2 = 3^3$, $a_3 = (3^3)^3$ and so on.)

(i) What digit appears in the unit place of a_7?

(ii) Show that $a_7 \geqslant 10^{100}$.

(iii) What is $\dfrac{a_7 + 1}{2a_7}$ correct to two places of decimals? Justify your answer.

4 Find all the solutions of the equation

$$|x+1| - |x| + 3|x-1| - 2|x-2| = x + 2.$$

5 Four rigid rods AB, BC, CD and DA are freely jointed together to form a quadrilateral in the plane. Show that if P, Q, R, S are the mid-points of the sides AB, BC, CD, DA, respectively, then

$$|AB|^2 + |CD|^2 + 2|PR|^2 = |AD|^2 + |BC|^2 + 2|QS|^2.$$

Deduce that $|PR|^2 - |QS|^2$ remains constant however the vertices move. (Here $|PR|$ denotes the length of PR.)

6 Find constants a_0, a_1, a_2, a_3, a_4, a_5, a_6 and b such that

$$x^4(1-x)^4 = (a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)(x^2+1) + b.$$

Hence, or otherwise, prove that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\mathrm{d}x = \frac{22}{7} - \pi.$$

Evaluate $\displaystyle\int_0^1 x^4(1-x)^4\mathrm{d}x$ and deduce that

$$\frac{22}{7} > \pi > \frac{22}{7} - \frac{1}{630}.$$

7 Find constants a_1, a_2, u_1 and u_2 such that, whenever P is a cubic polynomial,

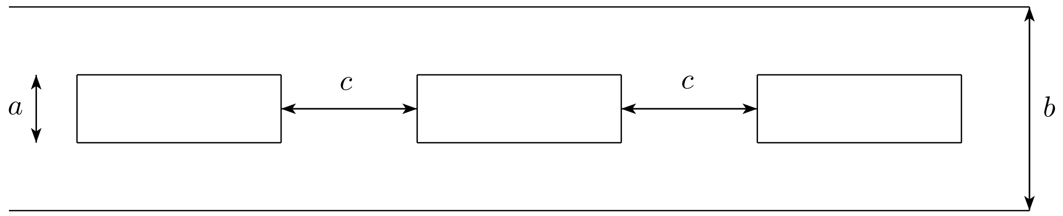
$$\int_{-1}^1 \mathrm{P}(t)\,\mathrm{d}t = a_1\mathrm{P}(u_1) + a_2\mathrm{P}(u_2).$$

8 By considering the maximum of $\ln x - x\ln a$, or otherwise, show that the equation $x = a^x$ has no real roots if $a > e^{1/e}$.

How many real roots does the equation have if $0 < a < 1$? Justify your answer.

Section B: Mechanics

9 A single stream of cars, each of width a and exactly in line, is passing along a straight road of breadth b with speed V. The distance between the successive cars is c.



A chicken crosses the road in safety at a constant speed u in a straight line making an angle θ with the direction of traffic. Show that

$$u \geqslant \frac{Va}{c\sin\theta + a\cos\theta}.$$

Show also that if the chicken chooses θ and u so that it crosses the road at the least possible uniform speed, it crosses in time

$$\frac{b}{V}\left(\frac{c}{a} + \frac{a}{c}\right).$$

10 The point A is vertically above the point B. A light inextensible string, with a smooth ring P of mass m threaded onto it, has its ends attached at A and B. The plane APB rotates about AB with constant angular velocity ω so that P describes a horizontal circle of radius r and the string is taut. The angle BAP has value θ and the angle ABP has value ϕ. Show that

$$\tan\frac{\phi - \theta}{2} = \frac{g}{r\omega^2}.$$

Find the tension in the string in terms of m, g, r, ω and $\sin\frac{1}{2}(\theta + \phi)$.

11 A particle of unit mass is projected vertically upwards in a medium whose resistance is k times the square of the velocity of the particle. If the initial velocity is u, prove that the velocity v after rising through a distance s satisfies

$$v^2 = u^2\mathrm{e}^{-2ks} + \frac{g}{k}(\mathrm{e}^{-2ks} - 1). \qquad (*)$$

Find an expression for the maximum height of the particle above the point of projection.

Does equation $(*)$ still hold on the downward path? Justify your answer.

Section C: Probability and Statistics

12 An experiment produces a random number T uniformly distributed on $[0,1]$. Let X be the larger root of the equation

$$x^2 + 2x + T = 0.$$

What is the probability that $X > -1/3$? Find $\mathrm{E}(X)$ and show that $\mathrm{Var}(X) = 1/18$.

The experiment is repeated independently 800 times generating the larger roots X_1, X_2, ..., X_{800}. If

$$Y = X_1 + X_2 + \cdots + X_{800}.$$

find an approximate value for K such that

$$\mathrm{P}(Y \leqslant K) = 0.08.$$

13 Mr Blond returns to his flat to find it in complete darkness. He knows that this means that one of four assassins Mr 1, Mr 2, Mr 3 or Mr 4 has set a trap for him. His trained instinct tells him that the probability that Mr i has set the trap is $i/10$. His knowledge of their habits tells him that Mr i uses a deadly trained silent anaconda with probability $(i+1)/10$, a bomb with probability $i/10$ and a vicious attack canary with probability $(9-2i)/10$ $[i = 1, 2, 3, 4]$.

He now listens carefully and, hearing no singing, concludes correctly that no canary is involved. If he switches on the light and the trap is a bomb he has probability $1/2$ of being killed but if the trap is an anaconda he has probability $2/3$ of survival. If he does not switch on the light and the trap is a bomb he is certain to survive but, if the trap is an anaconda, he has a probability $1/2$ of being killed. His professional pride means that he must enter the flat. Advise Mr Blond, giving reasons for your advice.

14 The maximum height X of flood water each year on a certain river is a random variable with density function

$$\mathrm{f}(x) = \begin{cases} \exp(-x) & \text{if } x \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

It costs y megadollars each year to prepare for flood water of height y or less. If $X \leqslant y$ no further costs are incurred but if $X \geqslant y$ the cost of flood damage is $r + s(X - y)$ megadollars where $r, s > 0$. The total cost T megadollars is thus given by

$$T = \begin{cases} y & \text{if } X \leqslant y, \\ y + r + s(X - y) & \text{if } X > y. \end{cases}$$

Show that we can minimise the expected total cost by taking

$$y = \ln(r + s).$$