Section A: Pure Mathematics

1 How many integers between $10\,000$ and $100\,000$ (inclusive) contain exactly two different digits? ($23\,332$ contains exactly two different digits but neither of $33\,333$ and $12\,331$ does.)

2 Show, by means of a suitable change of variable, or otherwise, that

$$\int_0^{\infty} \mathrm{f}((x^2+1)^{1/2}+x)\,\mathrm{d}x = \frac{1}{2}\int_1^{\infty}(1+t^{-2})\mathrm{f}(t)\,\mathrm{d}t.$$

Hence, or otherwise, show that

$$\int_0^{\infty}((x^2+1)^{1/2}+x)^{-3}\,\mathrm{d}x = \frac{3}{8}.$$

3 Which of the following statements are true and which are false? Justify your answers.

(i) $a^{\ln b} = b^{\ln a}$ for all $a, b > 0$.

(ii) $\cos(\sin\theta) = \sin(\cos\theta)$ for all real θ.

(iii) There exists a polynomial P such that $|\mathrm{P}(\theta) - \cos\theta| \leqslant 10^{-6}$ for all real θ.

(iv) $x^4 + 3 + x^{-4} \geqslant 5$ for all $x > 0$.

4 Prove that the rectangle of greatest perimeter which can be inscribed in a given circle is a square.

The result changes if, instead of maximising the sum of lengths of sides of the rectangle, we seek to maximise the sum of nth powers of the lengths of those sides for $n \geqslant 2$. What happens if $n = 2$? What happens if $n = 3$? Justify your answers.

5 **(i)** In the Argand diagram, the points Q and A represent the complex numbers $4 + 6i$ and $10 + 2i$. If A, B, C, D, E, F are the vertices, taken in clockwise order, of a regular hexagon (regular six-sided polygon) with centre Q, find the complex number which represents B.

(ii) Let a, b and c be real numbers. Find a condition of the form $Aa + Bb + Cc = 0$, where A, B and C are integers, which ensures that

$$\frac{a}{1+i} + \frac{b}{1+2i} + \frac{c}{1+3i}$$

is real.

6 Let $a_1 = \cos x$ with $0 < x < \pi/2$ and let $b_1 = 1$. Given that

$$\begin{aligned} a_{n+1} &= \tfrac{1}{2}(a_n + b_n), \\ b_{n+1} &= (a_{n+1}b_n)^{1/2}, \end{aligned}$$

find a_2 and b_2 and show that

$$a_3 = \cos\frac{x}{2}\cos^2\frac{x}{4} \quad \text{and} \quad b_3 = \cos\frac{x}{2}\cos\frac{x}{4}.$$

Guess general expressions for a_n and b_n (for $n \geqslant 2$) as products of cosines and verify that they satisfy the given equations.

7 My bank pays ρ% interest at the end of each year. I start with nothing in my account. Then for m years I deposit £a in my account at the beginning of each year. After the end of the mth year, I neither deposit nor withdraw for l years. Show that the total amount in my account at the end of this period is

$$£a\frac{r^{l+1}(r^m - 1)}{r - 1}$$

where $r = 1 + \dfrac{\rho}{100}$.

At the beginning of each of the n years following this period I withdraw £b and this leaves my account empty after the nth withdrawal. Find an expression for a/b in terms of r, l, m and n.

8 Fluid flows steadily under a constant pressure gradient along a straight tube of circular cross-section of radius a. The velocity v of a particle of the fluid is parallel to the axis of the tube and depends only on the distance r from the axis. The equation satisfied by v is

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v}{\mathrm{d}r}\right) = -k,$$

where k is constant. Find the general solution for v.

Show that $|v| \to \infty$ as $r \to 0$ unless one of the constants in your solution is chosen to be 0. Suppose that this constant is, in fact, 0 and that $v = 0$ when $r = a$. Find v in terms of k, a and r.

The volume F flowing through the tube per unit time is given by

$$F = 2\pi \int_0^a rv\,\mathrm{d}r.$$

Find F.

Section B: Mechanics

9 Two small spheres A and B of equal mass m are suspended in contact by two light inextensible strings of equal length so that the strings are vertical and the line of centres is horizontal. The coefficient of restitution between the spheres is e. The sphere A is drawn aside through a very small distance in the plane of the strings and allowed to fall back and collide with the other sphere B, its speed on impact being u. Explain briefly why the succeeding collisions will all occur at the lowest point. (Hint: Consider the periods of the two pendulums involved.)

Show that the speed of sphere A immediately after the second impact is $\frac{1}{2}u(1+e^2)$ and find the speed, then, of sphere B.

10 A shell explodes on the surface of horizontal ground. Earth is scattered in all directions with varying velocities. Show that particles of earth with initial speed v landing a distance r from the centre of explosion will do so at times t given by

$$\tfrac{1}{2}g^2t^2 = v^2 \pm \surd(v^4 - g^2r^2).$$

Find an expression in terms of v, r and g for the greatest height reached by such particles.

11 Hank's Gold Mine has a very long vertical shaft of height l. A light chain of length l passes over a small smooth light fixed pulley at the top of the shaft. To one end of the chain is attached a bucket A of negligible mass and to the other a bucket B of mass m. The system is used to raise ore from the mine as follows. When bucket A is at the top it is filled with mass $2m$ of water and bucket B is filled with mass λm of ore, where $0 < \lambda < 1$. The buckets are then released, so that bucket A descends and bucket B ascends. When bucket B reaches the top both buckets are emptied and released, so that bucket B descends and bucket A ascends. The time to fill and empty the buckets is negligible. Find the time taken from the moment bucket A is released at the top until the first time it reaches the top again.

This process goes on for a very long time. Show that, if the greatest amount of ore is to be raised in that time, then λ must satisfy the condition $\mathrm{f}'(\lambda) = 0$ where

$$\mathrm{f}(\lambda) = \frac{\lambda(1-\lambda)^{1/2}}{(1-\lambda)^{1/2} + (3+\lambda)^{1/2}}.$$

Section C: Probability and Statistics

12 Suppose that a solution (X, Y, Z) of the equation

$$X + Y + Z = 20,$$

with X, Y and Z non-negative integers, is chosen at random (each such solution being equally likely). Are X and Y independent? Justify your answer.

Show that the probability that X is divisible by 5 is $5/21$. What is the probability that XYZ is divisible by 5?

13 I have a bag initially containing r red fruit pastilles (my favourites) and b fruit pastilles of other colours. From time to time I shake the bag thoroughly and remove a pastille at random. (It may be assumed that all pastilles have an equal chance of being selected.) If the pastille is red I eat it but otherwise I replace it in the bag. After n such drawings, I find that I have only eaten one pastille. Show that the probability that I ate it on my last drawing is

$$\frac{(r+b-1)^{n-1}}{(r+b)^n - (r+b-1)^n}.$$

14 To celebrate the opening of the financial year the finance minister of Genland flings a Slihing, a circular coin of radius a cm, where $0 < a < 1$, onto a large board divided into squares by two sets of parallel lines 2 cm apart. If the coin does not cross any line, or if the coin covers an intersection, the tax on yaks remains unchanged. Otherwise the tax is doubled. Show that, in order to raise most tax, the value of a should be

$$\left(1 + \frac{\pi}{4}\right)^{-1}.$$

If, indeed, $a = \left(1 + \frac{\pi}{4}\right)^{-1}$ and the tax on yaks is 1 Slihing per yak this year, show that its expected value after n years will have passed is

$$\left(\frac{8+\pi}{4+\pi}\right)^n.$$