Section A: Pure Mathematics

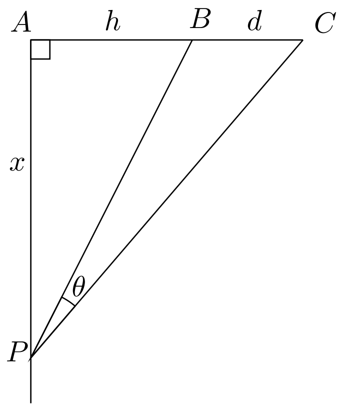
1 **(i)** The real numbers x and y satisfy the equation $4x^2 + 16xy + y^2 + 24x = 0$. Prove that either $x \leqslant 0$ or $x \geqslant \frac{2}{5}$, and, similarly, find restrictions on the values of y.

(ii) Find the coefficient of x^n in the expansion, in ascending powers of x, of

$$\frac{9}{(2-x)^2(1+x)}.$$

State the set of values of x for which this expansion is valid.

2 In the figure, the angle PAC is a right angle. $AB = h, BC = d, AP = x$ and the angle BPC is θ. Express $\tan\theta$ in terms of h, d and x.



Given that x may be varied, deduce, or find otherwise, the value of x (in terms of the constants d and h) for which θ has its maximum value.

3 Using the substitution $z = \dfrac{\mathrm{d}y}{\mathrm{d}x} - y$, or otherwise, solve the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 2\mathrm{e}^x,$$

given that

$$y = 1 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} = 2 \quad \text{when } x = 0.$$

4 Evaluate

$$\int_1^\infty \frac{1}{(x+1)\sqrt{x^2+2x-2}}\,\mathrm{d}x.$$

5 **(i)** You are writing down all the positive integers in increasing order, starting from 1, until you have written 1000 digits, at which point you stop (even possibly in the middle of a number). How many times have you used the digit 7? Explain your reasoning briefly but clearly.

(ii) In how many zeros does the number $365!$ terminate, when written in base 10? Justify your answer.

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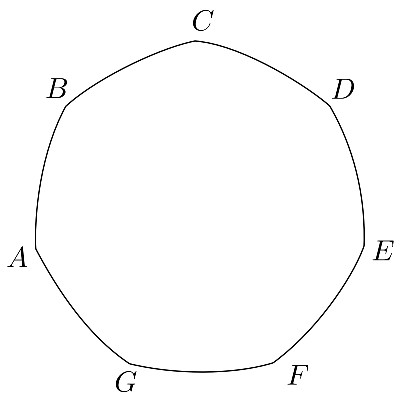

The diagram shows a cross-section, parallel to its faces, of a British 50 pence coin. The seven arcs $AB, BC, \ldots, FG, GA$ are all of equal length and each arc is formed from the circle having its centre at the vertex diametrically opposite the mid-point of the arc. Given that the radius of each of these circles is a, show that the area of a face of the coin is

$$\frac{a^2}{2}\left(\pi - 7\tan\frac{\pi}{14}\right).$$

7 Find the modulus and argument of $1+\mathrm{e}^{2\mathrm{i}\alpha}$ where $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$.

By using de Moivre's theorem, or otherwise, sum the series

$$\sum_{r=0}^{n} \frac{n!}{r!(n-r)!}\sin(2r+1)\alpha.$$

8 Let (a, b) be a fixed point, and (x, y) a variable point, on the curve $y = \mathrm{f}(x)$, $(x \geqslant a, \mathrm{f}'(x) \geqslant 0)$. The curve divides the rectangle with vertices $(a, b), (a, y), (x, y)$ and (x, b) into two portions, the lower of which has always half the area of the upper. Show that the curve is an arc of a parabola with its vertex at (a, b).

9 **(i)** Find the set of values of x which satisfy the inequality

$$\left| x + \frac{x-1}{x+1} \right| < 2,$$

expressing the set in terms of intervals whose end points are numbers of the form $a + b\sqrt{3}$, where a and b are integers.

(ii) Evaluate $\displaystyle\int_0^{\pi} |\cos x - \sin x| \ \mathrm{d}x.$

Section B: Mechanics

10 An engine of mass m moves along a straight level track against a resistance which at any instant has magnitude mkv, where v is the speed of the engine and k is a constant. The engine is working at a constant rate $9mku^2$. Given that the engine starts from rest, state the maximum possible speed of the engine, and explain briefly whether this speed can be obtained in a finite time. Show that the speed of the engine increases from u to $2u$ in time

$$\frac{1}{2k}\ln\left(\frac{8}{5}\right).$$

When the engine has attained a speed $2u$, the power is cut off, and an additional resistance to motion, of constant magnitude mkB, is applied so that the engine is brought to rest. Find the distance travelled by the engine with the power cut off.

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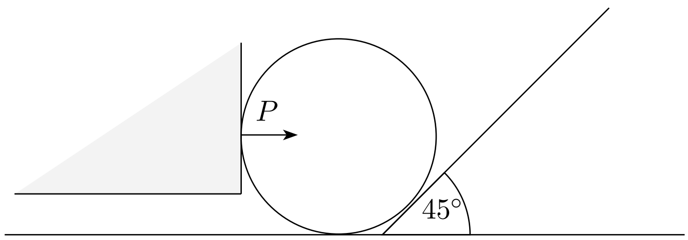

A uniform cylinder of mass M rests on a horizontal floor touching a loading ramp at 45° to the horizontal, as shown in the diagram. The cylinder is pushed from the side with a force of magnitude P by the vertical face of a piece of moving equipment. The coefficient of friction between the cylinder and the vertical face is μ and the coefficient of friction between the cylinder and the ramp is v. The value of P is such that the cylinder is just about to roll up the ramp. State on which surface the friction must be limiting, and hence show that

$$P = \frac{Mg}{1 - \mu(1+\sqrt{2})}.$$

Show further that $\mu < \sqrt{2} - 1$ and $v \geqslant \mu/(\sqrt{2} - \mu)$.

12 A game consists of a player sliding a small uniform disc A, initially at a point O, across horizontal ice. The disc collides with an identical disc B at rest at a distance d away from O. After the collision, B comes to rest at a target C distant $2d$ away from O. All motion is along the line OC and the discs may be treated as point masses. Show that the speed of B immediately after the collision is $\sqrt{2\mu g d}$, where μ is the coefficient of friction between the discs and the ice. Deduce that A is started with speed U given by

$$U = \frac{\sqrt{2\mu g d}}{1+e}(5+2e+e^2)^{\frac{1}{2}},$$

where e is the coefficient of restitution between the discs.

13 A particle of mass m is attached to one end of a light elastic string of natural length a, and the other end of the string is attached to a fixed point A. When at rest and hanging vertically the string has length $2a$. The particle is set in motion so that it moves in a horizontal circle below the level of A. The vertical plane through A containing the string rotates with constant angular speed ω. Show that for this motion to be possible, the string must be stretched to a length greater than $2a$ and ω must satisfy

$$\frac{g}{2a} < \omega^2 < \frac{g}{a}.$$

Section C: Probability and Statistics

14 A and B play the following game. A throws two unbiased four-faced dice onto a table (the four faces of each die are numbered $1, 2, 3, 4$ respectively). The total score is the sum of the numbers on the faces in contact with the table. B tries to guess this score, and guesses x. If his guess is right he wins $100x^2$ pence, and if his guess is wrong he loses $50x$ pence.

(i) Show that B's expected gain if he guesses 8 is 25 pence.

(ii) Which value of x would you advise B to choose, and what is his expected gain in this case?

15 The random variable C takes integral values in the interval -5 to 5, with probabilities

$$\mathrm{P}(C=-5)=\mathrm{P}(C=5)=\frac{1}{20},$$
$$\mathrm{P}(C=i)=\frac{1}{10}, \quad \text{for } -4 \leqslant i \leqslant 4.$$

Calculate the expectation and variance of C.

A shopper buys 36 items at random in a supermarket and, instead of adding up her bill exactly, she first rounds the cost of each item to the nearest 10 pence, rounding up or down with equal probability if there is an odd amount of 5 pence. Should she suspect a mistake if the cashier asks her for 20 pence more than she estimated? Explain your reasoning briefly but clearly.

16 The length in minutes of a telephone call made by a man from a public call-box is a random variable denoted by T. The probability density function of T is given by:

$$\mathrm{f}(t)=\begin{cases} 0 & t<0 \\ \frac{1}{2} & 0 \leqslant t<1, \\ k\mathrm{e}^{-2t} & t \geqslant 1, \end{cases}$$

where k is constant. Show that the expected length of a call is one minute. Find the cumulative distribution function of T.

To pay for a call, the man inserts into the coin box a 10 pence coin at the beginning of the call, and then another 10 pence coin after each half-minute of the call has elapsed. The random variable C is the cost in pence of the call. Prove that

$$\mathrm{E}(C)=22\tfrac{1}{2}+\frac{5}{\mathrm{e}-1}.$$

Comment *briefly* on why this value differs from the expected length of a call multiplied by the charging rate of 20 pence per minute.