Section A: Pure Mathematics

1 **(i)** Guess an expression for

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right),$$

valud for $n \geqslant 2$, and prove by mathematical induction the correctness of your guess.

(ii) Show that, if n is a positive integer,

$$\sum_{r=0}^{k}(-1)^r\binom{n}{r} = (-1)^k\binom{n-1}{k}, \qquad \text{for } k = 0, 1, \ldots, n-1.$$

2 Show by using the binomial expansion, or otherwise, that $(1+x)^n > nx$ whenever $x \geqslant 0$ and n is a positive integer. Deduce that if $y > 1$ then, given any number K, an integer N can be found such that $y^n > K$ for all integers $n \geqslant N$.

Show further that if $y > 1$ then, given any K, an integer N can be found such that $\dfrac{y^n}{n} > K$ for all integers $n \geqslant N$.

3 For the complex numbers z_1 and z_2 interpret geometrically the inequality

$$|z_1 + z_2| \leqslant |z_1| + |z_2|\,.$$

Prove that, if $|a_i| \leqslant 2$ for $i = 1, 2, \ldots, n$, then the equation

$$a_1 z + a_2 z^2 + \cdots + a_n z^n = 1$$

has no solutions with $|z| \leqslant \frac{1}{3}$.

4 Prove that the straight line

$$ty = x + at^2$$

touches the parabola $y^2 = 4ax$ and find the coordinates of the point of contact.

The tangents from a point to the parabola meet the directrix $(x = -a)$ in points L and M. Show that, if LM is of a fixed length ℓ, the point must lie on

$$(x+a)^2(y^2 - 4ax) = \ell^2 x^2.$$

5　The equation

$$\sin x = \lambda x, \qquad x \geqslant 0,$$

where $\lambda > 0$, has a finite number N of non-zero solutions x_n, $i = 1, \ldots, N$, where N depends on λ, provided $\lambda < 1$.

(i) Show by a graphical argument that there are no non-zero solutions for $\lambda > 1$. Show also that for $\lambda = 1 - \epsilon^2$, with $\epsilon > 0$ and very small compared to 1, there is a non-zero solution approximately equal to $\epsilon\sqrt{6}$.

(ii) Suppose that $N = 2R + 1$ where R is an integer, and that $x_1 < x_2 < \cdots < x_{2R+1}$. By drawing an appropriate graph, explain why

$$\begin{aligned} (2n-2)\pi < x_{2n-1} &< (2n-1)\pi \quad \text{for } n = 1, \ldots, R+1, \\ 2n\pi < x_{2n} &< (2n+\tfrac{1}{2})\pi \quad \text{for } n = 1, \ldots, R. \end{aligned}$$

Hence derive an approximate value for N in terms of λ, when λ is very small.

6　Let

$$I_n = \int_0^\infty \operatorname{sech}^n u \, \mathrm{d}u.$$

Show that for $n > 0$

$$\int_0^\infty \operatorname{sech}^{n+2} u \, \sinh^2 u \, \mathrm{d}u = \frac{1}{n+1} I_n,$$

and deduce that

$$(n+1) I_{n+2} = n I_n.$$

Find the value of I_6.

7　Show that the differential equation

$$x^2 y'' + (x-2)(xy' - y) = 0$$

has a solution proportional to x^α for some α. By making the substitution $y = x^\alpha v$, or otherwise, find the general solution of this equation.

8 **(i)** Using vectors, or otherwise, prove that the sum of the squares of the edges of any tetrahedron equals four times the sum of the squares of the lines joining the midpoints of opposite edges.

(ii) By using the inequality

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}|\,|\mathbf{b}|$$

with a suitable choices of three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$, or otherwise, prove that

$$3x + 2y + 6 \leqslant 7\sqrt{x^2 + y^2 + 1}.$$

For what values of x and y does the equality sign hold?

9 Prove that the set of all matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix},$$

where x, y and z are real numbers, is a group G under matrix multiplication. (You may assume that matrix multiplication is associative.)

Does the subset consisting of these matrices where x, y, z are restricted to the integers form a subgroup of G? Is there an element $\mathbf{A}$ in G, with $\mathbf{A}$ not equal to the identity matrix, such that $\mathbf{AB} = \mathbf{BA}$ for all $\mathbf{B}$ belonging to G? Justify your answer.

10 **(i)** Show that every odd square leaves remainder 1 when divided by 8, and that every even square leaves remainder 0 or 4. Deduce that a number of the form $8n + 7$, where n is a positive integer, cannot be expressed as a sum of three squares.

(ii) Prove that 17 divides $2^{3n+1} + 3(5^{2n+1})$ for all integers $n \geqslant 0$.

Section B: Mechanics

11 Two identical snowploughs plough the same stretch of road in the same direction. The first starts at time $t = 0$ when the depth of snow is h metres, and the second starts from the same point T seconds later. Snow falls so that the depth of snow increases at a constant rate of k ms^{-1}. It may be assumed that each snowplough moves at a speed equal to $k/(\alpha z)$ ms^{-1} with α a constant, where z is the depth of snow it is ploughing, and that it clears all the snow.

(i) Show that the time taken for the first snowplough to travel x metres is

$$(\mathrm{e}^{\alpha x} - 1)\frac{h}{k} \text{ seconds.}$$

(ii) Show that at time $t > T$, the second snowplough has moved y metres, where t satisfies

$$\frac{1}{\alpha}\frac{\mathrm{d}t}{\mathrm{d}y} = t - (\mathrm{e}^{\alpha y} - 1)\,\frac{h}{k}.$$

Verify that the required solution of this equation is

$$t = (\mathrm{e}^{\alpha y} - 1)\,\frac{h}{k} + \left(T - \frac{\alpha h y}{k}\right)\mathrm{e}^{\alpha y}$$

and deduce that the snowploughs collide when they have moved a distance $kT/(\alpha h)$ metres.

12 One end A of a uniform straight rod AB of mass M and length L rests against a smooth vertical wall. The other end B is attached to a light inextensible string BC of length αL which is fixed to the wall at a point C vertically above A. The rod is in equilibrium with the points A, B and C not collinear. Determine the inclination of the rod to the vertical and the set of possible values of α.

Show that the tension in the string is

$$\frac{Mg\alpha}{2}\left(\frac{3}{\alpha^2 - 1}\right)^{\frac{1}{2}}.$$

13 A particle of mass m is attached to a light circular hoop of radius a which is free to roll in a vertical plane on a rough horizontal table. Initially the hoop stands with the particle at its highest point and is then displaced slightly. Show that while the hoop is rolling on the table, the speed v of the particle when the radius to the particle makes an angle 2θ with the upward vertical is given by

$$v = 2(ga)^{\frac{1}{2}} \sin\theta.$$

Write down expressions in terms of θ for x, the horizontal displacement of the particle from its initial position, and y, its height above the table, and use them to show that

$$\theta = \tfrac{1}{2}(g/a)^{\frac{1}{2}} \tan\theta$$

and

$$\ddot{y} = -2g\sin^2\theta.$$

By considering the reaction of the table on the hoop, or otherwise, describe what happens to prevent the hoop rolling beyond the position for which $\theta = \pi/4$.

14 A uniform straight rod of mass m and length $4a$ can rotate freely about its midpoint on a smooth horizontal table. Initially the rod is at rest. A particle of mass m travelling on the table with speed u at right angles to the rod collides perfectly elastically with the rod at a distance a from the centre of the rod. Show that the angular speed, ω, of the rod after the collision is given by

$$a\omega = 6u/7.$$

Show also that the particle and rod undergo a subsequent collision.

Section C: Probability and Statistics

15 The mountain villages A, B, C and D lie at the vertices of a tetrahedron, and each pair of villages is joined by a road. After a snowfall the probability that any road is blocked is p, and is independent of the conditions on any other road. Find the probability that it is possible to travel from any village to any other village by some route after a snowfall. In the case $p = \frac{1}{2}$ show that this probability is $19/32$.

16 The new president of the republic has no children. According to custom, he decides on accession that he will have no more children once he has r sons. It may be assumed that each baby born to him is equally likely to be a boy or a girl, irrespective of the sexes of his previous children. Let C be the final number of children in his family.

(i) Find, in terms of r, the expectation of C.

(ii) By considering the numbers of boys and girls among $2r-1$ children, or otherwise, show that $\mathrm{P}(C < 2r) = \frac{1}{2}$.